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Conformal topological Yang-Mills theory and de Sitter holography

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Abstract: A new topological conformal field theory in four euclidean dimensions is constructed from $\mathcal{N} = 4$ super Yang-Mills theory by twisting the whole of the conformal group with the whole of the R-symmetry group, resulting in a theory that is conformally invariant and has two conformally invariant BRST operators. A curved space generalisation is found on any riemannian 4-fold. This formulation has local Weyl invariance and two Weyl-invariant BRST symmetries, with an action and energy-momentum tensor that are BRST-exact. This theory is expected to have a holographic dual in 5-dimensional de Sitter space.

Keywords: Topological Field Theories, Duality in Gauge Field Theories.

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Contents

1. Introduction

In [1] it was proposed that theories in D-dimensional de Sitter space could have a holographic dual which is a conformal field theory in $D-1$ dimensional euclidean space. In particular, in [1, 2] supersymmetric theories were identified for which an argument analogous to that of [3] for anti-de Sitter holography could be made. These included a five dimensional de Sitter vacuum arising from a solution of the type IIB* string theory. This solution preserves 32 supersymmetries and also arises as a solution of a five dimensional gauged supergravity. The proposed dual theory is the $\mathcal{N} = 4$ superconformal Yang-Mills theory in four euclidean dimensions. Unfortunately, both the $D = 5$ supergravity and the $D = 4$ super-Yang-Mills theories are non-standard in that they have some fields with kinetic terms of the wrong sign. However, it was pointed out in [1] that the super-Yang-Mills theory that arises in this way is precisely the one that can be twisted to obtain a topological field theory. This should correspond to a twisting of the five dimensional supergravity theory, so that the 't Hooft limit of the topological field theory should have a dual description as a topological supergravity theory in five dimensional de Sitter space.

In four euclidean dimensions, the Lorentz group is $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. In the usual twistings, Spin(4) or an SU(2) subgroup thereof is identified with a subgroup of the R-symmetry group. In the $\mathcal{N} = 4$ theories, this necessarily breaks the R-symmetry group. It was pointed out in [1] that the R-symmetry group is in fact the same as the conformal group in such theories, so there is the possibility of twisting the whole of the R-symmetry group with the conformal group, and this seems the most natural twisting to use in the holographic context. The purpose of this paper is to further develop these proposals, and in particular to construct a new topological conformal field theory in which the conformal group is twisted with the whole of the R-symmetry group.

It will be useful to begin by reviewing two ways of viewing topological field theories. Consider first the twisting of $\mathcal{N}=2$ supersymmetric Yang-Mills to give a topological theory whose observables are the Donaldson invariants [4]. The $\mathcal{N}=2$ supersymmetric Yang-Mills theory in 3+1 dimensions can be obtained by reducing $D=6$, $\mathcal{N}=2$ supersymmetric Yang-Mills theory on a spatial 2-torus. The $D=6$ theory has an SU(2) R-symmetry and the reduction gives a theory with U(2) R-symmetry, the extra SO(2) being related to rotations in the 5-6 plane. (This SO(2) would be broken if one were to keep the massive Kaluza-Klein modes.) The bosonic sector is Lie-algebra valued and consists of a vector field and two scalars, which transform as a 2-vector under the SO(2). The Wick rotated version of this theory is usually taken to be a theory in four euclidean dimensions with U(2) R-symmetry and the same bosonic sector. There are the usual subtleties as to how one treats the fermions, but however one proceeds, the resulting theory in four euclidean dimensions has no conventional supersymmetry.

However, there is a simple way of obtaining a supersymmetric theory in four euclidean dimensions. One can simply start with the $D=6$ theory and reduce on one space and one time dimension, and the resulting theory in four euclidean dimensions will automatically be invariant under $\mathcal{N}=2$ supersymmetry [5, 6]. The R-symmetry of this theory is $\mathrm{SU}(2)\times\mathrm{SO}(1,1)$ and one of the scalar fields (the one arising from the time component of the $D=6$ vector field) has a kinetic term of the wrong sign; this sign is necessary for the non-compact R-symmetry and for invariance under euclidean supersymmetry. Following [1], it will be convenient to refer to this as a *euclidean field theory* and to the Wick-rotated one as a *euclideanised field theory.*

The twisting of [4] was originally formulated in terms of the euclideanised theory, with symmetry $\mathrm{Spin}(4)\times\mathrm{U}(2)\cong\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$. An SU(2) subgroup of the Spin(4) Lorentz symmetry is twisted with the SU(2) subgroup of the U(2) R-symmetry, so that one of the supercharges becomes a scalar BRST charge. Equivalently, different twistings correspond to regarding different embeddings of $\mathrm{Spin}(4)\cong\mathrm{SU}(2)\times\mathrm{SU}(2)$ in $\mathrm{Spin}(4)\times\mathrm{SU}(2)\cong\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ as the Lorentz symmetry group. However, in this approach, there are some subtleties in finding the action invariant under the twisted supersymmetry, and in particular, the sign of the kinetic term of one of the two original scalars must be changed, so that the twisted theory has an $\mathrm{SO}(1,1)$ symmetry instead of the original SO(2) [4]; this $\mathrm{SO}(1,1)$ symmetry is the ghost-number symmetry, with the charge of a field being its ghost-number. In performing the functional integral, the problematic scalar is usually analytically continued $\phi\to i\phi$. The negative-norm states corresponding to this field are not in the BRST cohomology and in this sense are not physical.

However, this topological field theory with $\mathrm{SO}(1,1)$ ghost-number symmetry can be constructed directly by twisting the euclidean $\mathcal{N}=2$ supersymmetric Yang-Mills theory [5, 6]. In this case the twisting of an SU(2) subgroup of the Spin(4) with the SU(2) subgroup of the $\mathrm{SU}(2)\times\mathrm{SO}(1,1)$ R-symmetry is straightforward and automatically gives a theory invariant under twisted supersymmetry, and the $\mathrm{SO}(1,1)$ ghost number symmetry is inherited directly from the euclidean theory. For the euclideanised theory, the extra sign changes needed to obtain twisted supersymmetry corresponded in the untwisted theory to changing the non-supersymmetric euclideanised theory with $\mathrm{SU}(2) \times \mathrm{SO}(2)$ R-symmetry to the supersymmetric euclidean one with $\mathrm{SU}(2) \times \mathrm{SO}(1,1)$ R-symmetry. The construction from twisting the euclidean theory is clearly the more economical and straightforward.

The situation is similar for the $\mathcal{N} = 4$ theories. Reducing supersymmetric Yang-Mills from $9+1$ to 4 dimensions can give either a lorentzian theory in $3+1$ dimensions with $\mathrm{Spin}(6)$ R-symmetry or to a theory in four euclidean dimensions with $\mathrm{Spin}(5,1)$ R-symmetry. Both theories have a vector and six scalars, which transform as a 6-vector under the R-symmetry, and both are invariant under $\mathcal{N} = 4$ supersymmetry. Wick rotating the lorentzian theory gives the euclideanised theory with $\mathrm{Spin}(6)$ R-symmetry and no conventional supersymmetry. Both the euclidean and the euclideanised theories have 6 scalars, but in the euclidean theory, one of the scalars has a kinetic term of the wrong sign.

The corresponding topological field theories were constructed from the twisting of the $\mathrm{Spin}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ Lorentz symmetry of the euclideanised theory with a $\mathrm{Spin}(4) \cong \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ subgroup of the R-symmetry, embedded as

$$\mathrm{Spin}(4) \times \mathrm{Spin}(2) \subset \mathrm{Spin}(6)\,. \tag{1.1}$$

There are three inequivalent topological twistings [7, 8, 9, 10]. In the half-twisted model, one twists $\mathrm{SU}(2)_L$ by $\mathrm{SU}(2)_1$, in the B-model or diagonal twisting one in addition twists $\mathrm{SU}(2)_R$ by $\mathrm{SU}(2)_2$, while in the A-model one twists $\mathrm{SU}(2)_L$ by the diagonal subgroup $\mathrm{SU}(2)_D$ of $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$. In constructing the invariant twisted action, it is necessary to change some signs and in particular the sign of the kinetic term of one of the scalars, so that the $\mathrm{SO}(2)$ R-symmetry in (1.1) becomes an $\mathrm{SO}(1,1)$, which is the ghost-number symmetry.

Again, the twistings can be constructed more directly from the euclidean theory, which automatically generates a theory with twisted supersymmetry without needing to add further sign changes by hand. In this case, the Lorentz symmetry is twisted with the $\mathrm{Spin}(4)$ subgroup embedded in the R-symmetry group as

$$\mathrm{Spin}(4) \times \mathrm{Spin}(1,1) \subset \mathrm{Spin}(5,1) \tag{1.2}$$

in one of the three inequivalent ways described above.

There are then three related Yang-Mills theories in four dimensions, the lorentzian one, the euclidean one and the euclideanised one. They have Lorentz and R-symmetries given respectively by

$$\mathrm{Spin}(3,1) \times \mathrm{Spin}(6)\,, \qquad \mathrm{Spin}(4) \times \mathrm{Spin}(5,1)\,, \qquad \mathrm{Spin}(4) \times \mathrm{Spin}(6)\,. \tag{1.3}$$

The first two have 16 supersymmetries, while the euclideanised one has none. Each of these theories is in fact conformally invariant, so that the Lorentz group is enlarged to the conformal group, and the three theories have bosonic symmetries given by

$$\mathrm{Spin}(4,2) \times \mathrm{Spin}(6)\,, \qquad \mathrm{Spin}(5,1) \times \mathrm{Spin}(5,1)\,, \qquad \mathrm{Spin}(5,1) \times \mathrm{Spin}(6)\,, \tag{1.4}$$

respectively. The lorentzian and euclidean theories are in fact superconformally invariant, with conformal supergroups $\mathrm{SU}(2,2|4)$ and $\mathrm{SU}^*(4|4)$ respectively.

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The three twistings described above all explicitly break the R-symmetry. However, when viewed as a twisting of the euclidean theory, there is in addition a fourth possible twisting [1]. The euclidean theory has symmetry $\mathrm{Spin}(5,1)\times\mathrm{Spin}(5,1)$ and there is the possibility of twisting the whole of the conformal symmetry with the whole of the R-symmetry group, giving a theory manifestly invariant under the diagonal $\mathrm{Spin}(5,1)$ subgroup. One of our aims here is to present this conformal twisting. There are some subtleties arising as the conformal group is non-linearly realised. Our approach will be to start with the B-model in which the Spin(4) Lorentz symmetry is twisted with a Spin(4) subgroup of the R-symmetry. This theory is however not invariant under the full conformal group [9] — it is invariant under dilatations but not under special conformal transformations. We will find modifications of the usual special conformal transformations that are a symmetry of the SO(4) twisted theory, and use these to construct further twistings, resulting in a topological conformal field theory with a BRST charge and an anti-BRST charge, both of which are invariant under the twisted conformal group.

This paper is organised as follows. In section 2 we will discuss the group theory behind the conformal twisting of the euclidean $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We also discuss the topological conformal field theories arising from twisted $\mathcal{N} = 2$ supersymmetric theories at conformal fixed points. In section 3 we discuss the SO(4) twisted euclidean $\mathcal{N} = 4$ theory and its symmetries, and in particular find new modifications of the standard special conformal transformations that are a symmetry of the theory. We find linear combinations of (modified) conformal transformations and R-symmetries that constitute the twisted $\mathrm{Spin}(5,1)$ symmetry of this theory. These twisted conformal generators do not commute with the scalar supercharges, but rather yield extra conformal supercharges. We will then show that two linear combinations of the supercharges are conformally invariant and define an anticommuting pair of conformally invariant BRST operators. The conformal symmetry in the twisted theory does not act in the standard way. We partially resolve this in section 4 by redefining the fields in such a way that the conformal transformations take a more standard form, giving a new form of the action invariant under the twisted conformal group and BRST charges.

In section 5, we construct the conserved BRST-exact symmetric traceless energy-momentum tensor of the flat space twisted theory. Then, in section 6 we couple the theory to gravity, adding non-minimal terms to the minimally-coupled action to obtain a theory which is invariant under Weyl rescalings and two BRST symmetries, with an action that is BRST-exact. This allows us to briefly discuss the topological invariants arising as observables in the theory.

In section 7 we discuss the theta-term in the action and S-duality. Finally, in section 8 we discuss some of the implications of our results to the arguments of [1] that such a topological conformal field theory should have a holographic description as a theory in 5-dimensional de Sitter space.

We also mention here some other work, not directly related to ours, that has been done on topological field theories and holography in recent years. A version of topological holography in three dimensions has been developed. On the gauge theory side this involves Chern-Simons theory, describing knot and three manifold invariants. There is a description

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of this theory using open topological A-strings ending on a lagrangian submanifold of a Calabi-Yau threefold [11]. In [12] it was proposed for the case of the three sphere that there is a dual formulation of this Chern-Simons theory based on closed topological A strings on the resolved conifold. This proposal has since been elaborated and extended (see [13] and references therein). A four dimensional topological field theory and its possible holographic dual has also been discussed in [14].

2. Twisting and group theory

Dimensional reduction of (9+1)-dimensional supersymmetric Yang-Mills from $\mathbb{R}^{9,1}$ to $\mathbb{R}^4$ gives the four dimensional euclidean $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The ten-dimensional theory has sixteen supercharges in a real chiral representation $\underline{\mathbf{16}}$ of the spin group $\mathrm{Spin}(9,1)$. After dimensional reduction, the R-symmetry group is $\mathrm{Spin}(5,1)$, which is isomorphic to $\mathrm{SU}^*(4) \cong \mathrm{SL}(2,\mathbb{H})$, with right and left handed complex Weyl spinor representations of complex dimension 4, which we will denote $\underline{\mathbf{4}}$ and $\underline{\mathbf{4'}}$ respectively. Under $\mathrm{Spin}(9,1) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}^*(4)$, the spinors decompose as

$$\underline{\mathbf{16}} \to (\underline{\mathbf{2}}, \underline{\mathbf{1}}, \underline{\mathbf{4'}}) \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{4}}). \tag{2.1}$$

We introduce indices $I, J = 1, \ldots, 4$ labelling the $\underline{\mathbf{4}}$ of $\mathrm{SU}^*(4)$, $I', J' = 1, \ldots, 4$ for the $\underline{\mathbf{4'}}$ of $\mathrm{SU}^*(4)$, $A, B = 1, 2$ labelling the $\underline{\mathbf{2}}$ of the first $\mathrm{SU}(2)$ and $\dot{A}, \dot{B} = 1, 2$ for the $\underline{\mathbf{2}}$ of the second $\mathrm{SU}(2)$. All of these indices are raised or lowered by complex conjugation,

One must take the underlying real representation of a complex representation R with a real structure: the Majorana condition on a spinor in the $\underline{\mathbf{16}}$ of $\mathrm{Spin}(9,1)$ becomes a symplectic Majorana condition in four dimensions. For example, the spinor $\lambda_{AI'}$ in the $(\underline{\mathbf{2}}, \underline{\mathbf{1}}, \underline{\mathbf{4'}})$ arising from the reduction of a Majorana-Weyl fermion in 10 dimensions satisfies the reality condition

$$(\lambda^*)^{AI'} = \epsilon^{AB} \Omega^{I'J'} \lambda_{BJ'}, \tag{2.2}$$

where $\Omega^{I'J'}$ is the symplectic invariant of $\mathrm{SU}^*(4)$ (which can be thought of as the charge conjugation matrix in 5+1 dimensions).

On dimensional reduction, the ten-dimensional super-Poincaré generators M_{MN}, P_M ($M, N = 1, \ldots 10$) and Q decompose into the generators M_{mn} and P_m ($m, n = 1, \ldots, 4$) of the four-dimensional euclidean group $\mathrm{ISO}(4)$, the $\mathrm{SO}(5,1)$ R-symmetry generators $\mathsf{R}_{\mu\nu}$ (antisymmetric in μ, ν, with $\mu, \nu = 1, \ldots, 6$), and the supercharges $\mathsf{Q}_{I'A}$ and $\mathsf{Q}_{I\dot{A}}$. The supercharge $\mathsf{Q}_{I'A}$ transforms in the $(\underline{\mathbf{2}}, \underline{\mathbf{1}}, \underline{\mathbf{4'}})$ representation of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}^*(4)$, whilst $\mathsf{Q}_{I\dot{A}}$ transforms in the $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{4}})$.

The four-dimensional theory is superconformally invariant, with superconformal group $\mathrm{SU}^*(4|4)$ (a different real form of $\mathrm{SU}(2,2|4)$) generated by the super-Poincaré generators together with the dilatation D, the special conformal generator K_m and the conformal supercharges $\mathsf{S}_{I'\dot{A}}$ and S_{IA}. The bosonic subgroup is $\mathrm{Spin}(5,1) \times \mathrm{Spin}(5,1) \cong \mathrm{SU}^*(4) \times \mathrm{SU}^*(4)$, the product of the euclidean conformal group and the R-symmetry. The 32 (conformal) supercharges transform in the $(\underline{\mathbf{4'}}, \underline{\mathbf{4}}) \oplus (\underline{\mathbf{4}}, \underline{\mathbf{4'}})$ representation of $\mathrm{SU}^*(4) \times \mathrm{SU}^*(4)$, with a symplectic Majorana condition using the symplectic invariants of both factors.

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The *conformal* twisting is the diagonal embedding $g \mapsto (g, g)$ of the conformal group $\mathrm{SU}^*(4)$ in the bosonic symmetry $\mathrm{SU}^*(4) \times \mathrm{SU}^*(4)$. Under the diagonal embedding, we have

$$[(\underline{\mathbf{4'}}, \underline{\mathbf{4}})] \to \underline{\mathbf{15}} \oplus \underline{\mathbf{1}} \qquad \text{and} \qquad [(\underline{\mathbf{4}}, \underline{\mathbf{4'}})] \to \underline{\mathbf{15}} \oplus \underline{\mathbf{1}}\,, \tag{2.3}$$

yielding two scalar supercharges. The bosonic generators of $\mathrm{SU}^*(4) \times \mathrm{SU}^*(4)$ give under the embedding $(\underline{\mathbf{15}}, \underline{\mathbf{1}}) \to \underline{\mathbf{15}}$ and $(\underline{\mathbf{1}}, \underline{\mathbf{15}}) \to \underline{\mathbf{15}}$. The bracket of any two fermionic scalar supercharges must be a scalar bosonic generator. As there are none, we conclude that the scalar supercharges anticommute with each other and each squares to zero. In other words, the twisted theory has two (anticommuting) BRST operators. After the twisting, one has symmetry under the diagonal subgroup $\mathrm{SU}^*(4)_D$, which we will refer to as the twisted conformal group.

The conformal group $\mathrm{SU}^*(4)$ is non-linearly realised, but there is a subgroup CSpin(4) $:= \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}^+ \subset \mathrm{SU}^*(4)$, generated by the spin group and the dilatations, which is a manifest symmetry of the euclidean $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In particular, all fields transform as irreducible representations of $\mathrm{CSpin}(4) \times \mathrm{SU}^*(4)$.

The generators of the original superconformal algebra are

$$\{\mathsf{P}_m, \mathsf{M}_{mn}, \mathsf{D}, \mathsf{K}_m, \mathsf{R}_{\mu\nu}, \mathsf{Q}_{I'A}, \mathsf{Q}_{I\dot{A}}, \mathsf{S}_{IA}, \mathsf{S}_{I'\dot{A}}\}\,, \tag{2.4}$$

transforming in the following representation of $G = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}^+ \times \mathrm{SU}^*(4)$:

$$(\underline{\mathbf{2}}, \underline{\mathbf{2}}, \underline{\mathbf{1}})^{+2} \oplus (\underline{\mathbf{3}}, \underline{\mathbf{1}}, \underline{\mathbf{1}})^0 \oplus (\underline{\mathbf{1}}, \underline{\mathbf{3}}, \underline{\mathbf{1}})^0 \oplus (\underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{1}})^0 \oplus (\underline{\mathbf{2}}, \underline{\mathbf{2}}, \underline{\mathbf{1}})^{-2}$$
$$\oplus (\underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{15}})^0 \oplus (\underline{\mathbf{2}}, \underline{\mathbf{1}}, \underline{\mathbf{4}})^{-1} \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{4'}})^{-1} \oplus (\underline{\mathbf{2}}, \underline{\mathbf{1}}, \underline{\mathbf{4'}})^{+1} \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{4}})^{+1}\,,$$

with the superscript indicating the $\mathbb{R}^+$ conformal grading. In turn, the generators $\mathsf{R}_{\mu\nu}$ of the $\mathrm{SU}^*(4)$ R-symmetry transform as the $\underline{\mathbf{15}}$ of $\mathrm{SU}^*(4)$, which breaks into the following representations of the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}^+$:

$$(\underline{\mathbf{2}}, \underline{\mathbf{2}})^{+2} \oplus (\underline{\mathbf{3}}, \underline{\mathbf{1}})^0 \oplus (\underline{\mathbf{1}}, \underline{\mathbf{1}})^0 \oplus (\underline{\mathbf{1}}, \underline{\mathbf{3}})^0 \oplus (\underline{\mathbf{2}}, \underline{\mathbf{2}})^{-2}\,. \tag{2.5}$$

Introducing indices $m', n' = 1, \ldots, 4$ for the SO(4) subgroup of SO(5,1), the R-symmetry generators then decompose as

$$\mathsf{R}_{\mu\nu} \mapsto \{\mathsf{p}_{m'}, \mathsf{m}_{m'n'}, \mathsf{d}, \mathsf{k}_{m'}\}\,, \tag{2.6}$$

with the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}^+$ generated by $\mathsf{m}_{m'n'}$ and d. The $\mathbb{R}^+$ gradings of the generators $\{\mathsf{p}_{m'}, \mathsf{m}_{m'n'}, \mathsf{d}, \mathsf{k}_{m'}\}$ are thus $\{2, 0, 0, -2\}$.

The Spin(4) twisting to give the B-model of [9] is achieved by twisting the action of the rotation generators M_{mn} with the action of the R-symmetry generators $\mathsf{m}_{m'n'}$, so that the resulting theory is manifestly invariant under the new spin generators defined by

$$\mathbb{M}_{mn} \equiv \mathsf{M}_{mn} + \mathsf{m}_{mn}\,, \tag{2.7}$$

with the indices m, n identified with m', n'. This can then be enhanced to a CSpin(4) twisting by twisting the action of the dilatation D with the action of the d ghost number

$\mathbb{R}^+$ symmetry, to obtain a twisted dilatation generator

$$\mathbb{D} \equiv \mathsf{D} + \mathsf{d}\,. \tag{2.8}$$

Then the twisted conformal weight of a field is the sum of the conformal weight (defined as half the $\mathbb{R}^+$ conformal grading) and ghost number.

To obtain the full conformal twisting, one must in addition twist the momenta P_m and the special conformal generators K_m by the appropriate R-symmetry generators, so that the twisted conformal generators are

$$\begin{aligned} \mathbb{P}_m &\equiv \mathsf{P}_m + \mathsf{p}_m \\ \mathbb{M}_{mn} &\equiv \mathsf{M}_{mn} + \mathsf{m}_{mn} \\ \mathbb{D} &\equiv \mathsf{D} + \mathsf{d} \\ \mathbb{K}_m &\equiv \mathsf{K}_m + \mathsf{k}_m\,. \end{aligned} \tag{2.9}$$

Under the CSpin(4) twist, the spinor supercharges, which transform as the $(\underline{\mathbf{2}},\underline{\mathbf{1}},\underline{\mathbf{4'}})^{+1}\oplus(\underline{\mathbf{1}},\underline{\mathbf{2}},\underline{\mathbf{4}})^{+1}$ of CSpin(4) $\times$ SU*(4), are twisted to generators in the following representations of the diagonal CSpin(4):

$$(\underline{\mathbf{2}},\underline{\mathbf{1}},\underline{\mathbf{4'}})^{+1}\oplus(\underline{\mathbf{1}},\underline{\mathbf{2}},\underline{\mathbf{4}})^{+1} \to (\underline{\mathbf{3}},\underline{\mathbf{1}})^{0}\oplus(\underline{\mathbf{1}},\underline{\mathbf{1}})^{0}\oplus(\underline{\mathbf{2}},\underline{\mathbf{2}})^{+2}\oplus(\underline{\mathbf{2}},\underline{\mathbf{2}})^{+2}\oplus(\underline{\mathbf{1}},\underline{\mathbf{3}})^{0}\oplus(\underline{\mathbf{1}},\underline{\mathbf{1}})^{0}. \tag{2.10}$$

This corresponds to the replacements $\{\mathsf{Q}_{I'A},\mathsf{Q}_{I\dot{A}}\}\to\{\mathsf{Q}^{\pm(0)}_{[mn]},\mathsf{Q}^{(0)},\tilde{\mathsf{Q}}^{(0)},\mathsf{Q}^{(+2)}_m,\tilde{\mathsf{Q}}^{(+2)}_m\}$ and similarly one has $\{\mathsf{S}_{I'\dot{A}},\mathsf{S}_{IA}\}\to\{\mathsf{S}^{\pm(0)}_{[mn]},\mathsf{S}^{(0)},\tilde{\mathsf{S}}^{(0)},\mathsf{S}^{(-2)}_m,\tilde{\mathsf{S}}^{(-2)}_m\}$. Using this then allows the original superconformal algebra to be decomposed in terms of brackets involving the twisted supercharges. We are primarily interested in the scalar supercharges here (i.e. the singlets $\mathsf{Q},\tilde{\mathsf{Q}},\mathsf{S},\tilde{\mathsf{S}}$) and their algebra is found to be (dropping the superscripts denoting the $\mathbb{R}^+$ conformal gradings)

$$\begin{aligned} [\mathsf{Q},\mathbb{P}_m] &= -\mathsf{Q}_m & [\tilde{\mathsf{Q}},\mathbb{P}_m] &= \tilde{\mathsf{Q}}_m \\ [\mathsf{Q},\mathbb{K}_m] &= -\tilde{\mathsf{S}}_m & [\tilde{\mathsf{Q}},\mathbb{K}_m] &= -\mathsf{S}_m \\ [\mathsf{S},\mathbb{P}_m] &= \tilde{\mathsf{Q}}_m & [\tilde{\mathsf{S}},\mathbb{P}_m] &= \mathsf{Q}_m \\ [\mathsf{S},\mathbb{K}_m] &= -\mathsf{S}_m & [\tilde{\mathsf{S}},\mathbb{K}_m] &= \tilde{\mathsf{S}}_m \\ [\mathsf{Q},\mathsf{S}] &= -4\mathbb{D} & [\tilde{\mathsf{Q}},\tilde{\mathsf{S}}] &= -4\mathbb{D} \end{aligned} \tag{2.11}$$

where we have only written down the non-zero brackets. The linear combinations

$$\mathbb{Q} := \mathsf{Q} + \tilde{\mathsf{S}}\,, \qquad \tilde{\mathbb{Q}} := \tilde{\mathsf{Q}} - \mathsf{S} \tag{2.12}$$

square to zero, anticommute with each other, and commute with the twisted conformal generators:

$$[\mathbb{Q},\mathbb{X}] = [\tilde{\mathbb{Q}},\mathbb{X}] = 0 \qquad \text{for any} \quad \mathbb{X}\in\{\mathbb{P}_m,\mathbb{M}_{mn},\mathbb{D},\mathbb{K}_m,\mathbb{Q},\tilde{\mathbb{Q}}\}\,. \tag{2.13}$$

The Spin(5,1) generators $\{\mathbb{P}_m,\mathbb{M}_{mn},\mathbb{D},\mathbb{K}_m\}$ satisfy the algebra

$$[\mathbb{M}_{mn},\mathbb{M}_{pq}] = \delta_{mp}\mathbb{M}_{nq} + \delta_{nq}\mathbb{M}_{mp} - \delta_{mq}\mathbb{M}_{np} - \delta_{np}\mathbb{M}_{mq}$$

$$
\begin{aligned}
[\mathbb{M}_{mn},\mathbb{P}_p] &= \delta_{mp}\mathbb{P}_n - \delta_{pn}\mathbb{P}_m \\
[\mathbb{M}_{mn},\mathbb{K}_p] &= \delta_{mp}\mathbb{K}_n - \delta_{pn}\mathbb{K}_m \\
[\mathbb{D},\mathbb{P}_m] &= \mathbb{P}_m \\
[\mathbb{D},\mathbb{K}_m] &= -\mathbb{K}_m \\
[\mathbb{K}_m,\mathbb{P}_n] &= 2\delta_{mn}\mathbb{D} + 2\mathbb{M}_{mn}\,,
\end{aligned}
\tag{2.14}
$$

with all other brackets vanishing.

It is interesting to ask how much of this structure survives for twisted $\mathcal{N}=2$ theories at conformal fixed points. Such theories have been investigated in [15, 16]. A twisted $\mathcal{N}=2$ theory is invariant under Poincaré symmetry, together with the symmetries generated by a BRST charge Q and a ghost number charge d. At a conformal fixed point, the theory is invariant under the conformal group, which is generated by the twisted Lorentz generators $\mathbb{M}_{mn}$ together with $\{\mathsf{P}_m,\mathsf{D},\mathsf{K}_m\}$, and also invariant under the ghost number symmetry generated by d, the BRST symmetry generated by Q and a conformal BRST generated by S, arising from twisting the conformal supersymmetry. These satisfy a subalgebra of the algebra discussed above, but without the generators $\tilde{\mathsf{Q}},\tilde{\mathsf{S}}$. In this case, there do not seem to be any further twistings that lead to conformally invariant BRST operators of the type discussed above.

3. The diagonally twisted $\mathcal{N}=4$ theory and conformal invariance

Before twisting, the fields of the euclidean $\mathcal{N}=4$ supersymmetric Yang-Mills theory are $A_m, \lambda_{IA}, \lambda_{I'\dot{A}}, \phi_{IJ}$, with ϕ antisymmetric in its indices. Consider first the diagonal twisting of the SO(4) Lorentz symmetry with an SO(4) subgroup of the R-symmetry to obtain the B-model. The vector field is an R-singlet and remains unchanged, while the 6 scalars ϕ_{IJ} twist to a vector V_m and two scalars B, C. The fermions λ twist to give the anticommuting fields $\psi_m, \tilde{\psi}_m, \chi^{\pm}_{mn}, \eta, \tilde{\eta}$, where $\chi^{\pm}_{mn}$ are 2-forms satisfying $\chi^{\pm}_{mn} = \pm * \chi^{\pm}_{mn}$. We define $(X_{[mn]})^{\pm} \equiv \frac{1}{2}(X_{[mn]} \pm *X_{[mn]})$ and $X_{[mn]} \equiv \frac{1}{2}(X_{mn} - X_{nm})$.

The Spin(4) twisted action is [9]:

$$
\begin{aligned}
\mathcal{S}^{(0)} = \frac{1}{e^2}\int d^4x\ \mathrm{Tr}\Big(&-\mathcal{D}_m B \mathcal{D}^m C - \mathcal{D}_m V_n \mathcal{D}^m V^n - \frac{1}{4}F_{mn}F^{mn} + \\
&+ \mathcal{D}_m\psi_n(4\chi^{+mn} - \delta^{mn}\eta) + \mathcal{D}_m\tilde{\psi}_n(4\chi^{-mn} - \delta^{mn}\tilde{\eta}) - \\
&- \frac{i}{8\sqrt{2}}\Big((4\chi^+_{mn} - \delta_{mn}\eta)[4\chi^{+mn} - \delta^{mn}\eta, C] + \\
&\qquad + (4\chi^-_{mn} - \delta_{mn}\tilde{\eta})[4\chi^{-mn} - \delta^{mn}\tilde{\eta}, C]\Big) - \\
&- i\sqrt{2}\left((4\chi^+_{mn} - \delta_{mn}\eta)[\tilde{\psi}^m, V^n] - (4\chi^-_{mn} - \delta_{mn}\tilde{\eta})[\psi^m, V^n]\right) + \\
&+ i\sqrt{2}\,(\psi_m[\psi^m, B] + \tilde{\psi}_m[\tilde{\psi}^m, B]) - \\
&- \frac{1}{2}[B,C]^2 + 2[B,V_m][C,V^m] + [V_m,V_n][V^m,V^n]\Big) - \\
&- \frac{i\theta}{32\pi^2}\int d^4x\ \mathrm{Tr}\, *F_{mn}F^{mn}\,.
\end{aligned}
\tag{3.1}
$$

We follow the notation in [17]. Also $\mathcal{D}_m := \partial_m + i[A_m, \,]$ is the gauge covariant derivative acting on the fields, which are valued in the adjoint representation of the Lie algebra of the non-abelian gauge symmetry group. The field strength is

$$F_{mn} = \partial_m A_n - \partial_n A_m + i[A_m, A_n]\,, \tag{3.2}$$

while e is the usual Yang-Mills coupling constant and θ is the theta-parameter, which will be discussed further in section 7.

The theory is invariant under the SO(1,1) scale transformations generated by the dilatation D, with each field having a conformal weight c, and under the SO(1,1) subgroup of the SO(5,1) R-symmetry group that commutes with the SO(4) that was twisted. This is generated by d and is usually referred to as the ghost-number symmetry, with the SO(1,1) weight of each field referred to as the ghost number g of that field. The fields

$$\{B, C, A_m, V_m; \psi_m, \tilde{\psi}_m, \chi^{\pm}_{mn}, \eta, \tilde{\eta}\} \tag{3.3}$$

have conformal weights and ghost numbers $\{(c, g)\}$ given by

$$\left\{(1,1),(1,-1),(1,0),(1,0);\left(\frac{3}{2},\frac{-1}{2}\right),\left(\frac{3}{2},\frac{-1}{2}\right),\left(\frac{3}{2},\frac{1}{2}\right),\left(\frac{3}{2},\frac{1}{2}\right),\left(\frac{3}{2},\frac{1}{2}\right)\right\}, \tag{3.4}$$

respectively. The twisted dilatation was defined as $\mathbb{D} \equiv \mathsf{D} + \mathsf{d}$ and for a field with D, d weights $\{(c, g)\}$, the weight with respect to $\mathbb{D}$ is $c + g$. Then the twisted bosonic fields are

$$\{B^{(2)}, C^{(0)}, A^{(1)}_m, V^{(1)}_m\}, \tag{3.5}$$

and the fermionic fields are

$$\{\psi^{(1)}_m, \tilde{\psi}^{(1)}_m, \chi^{(2)\pm}_{mn}, \eta^{(2)}, \tilde{\eta}^{(2)}\}, \tag{3.6}$$

where the superscripts (suppressed in the following) give the twisted conformal weight $c+g$. The twisted theory is automatically invariant under the twisted scale transformations generated by $\mathbb{D}$ and so can be regarded as a twisting of CSpin(4) rather than just of Spin(4). The twisted dilatations act in the standard way, i.e., $\mathbb{D} \cdot \Phi = (x^n \partial_n + (c + g)_\Phi)\Phi$, with the weight $(c + g)_\Phi$ for each field Φ defined as above.

As discussed in the previous section, twisting produces various fermionic charges. In particular, we will be interested in the scalar BRST supercharges Q and $\tilde{\mathsf{Q}}$. The action of these on the fields is given by

$$\begin{aligned}
\mathsf{Q}\cdot A_m &= 2\psi_m & \tilde{\mathsf{Q}}\cdot A_m &= -2\tilde{\psi}_m \\
\mathsf{Q}\cdot \psi_m &= \sqrt{2}\,\mathcal{D}_m C & \tilde{\mathsf{Q}}\cdot \psi_m &= -2i[V_m, C] \\
\mathsf{Q}\cdot \tilde{\psi}_m &= -2i[V_m, C] & \tilde{\mathsf{Q}}\cdot \tilde{\psi}_m &= -\sqrt{2}\,\mathcal{D}_m C \\
\mathsf{Q}\cdot \chi^+_{mn} &= -F^+_{mn} + 2i[V_m, V_n]^+ & \tilde{\mathsf{Q}}\cdot \chi^+_{mn} &= 2\sqrt{2}\,(\mathcal{D}_{[m}V_{n]})^+ \\
\mathsf{Q}\cdot \chi^-_{mn} &= 2\sqrt{2}\,(\mathcal{D}_{[m}V_{n]})^- & \tilde{\mathsf{Q}}\cdot \chi^-_{mn} &= F^-_{mn} - 2i[V_m, V_n]^- \\
\mathsf{Q}\cdot \eta &= 2i[B, C] & \tilde{\mathsf{Q}}\cdot \eta &= -2\sqrt{2}\,\mathcal{D}_m V^m \\
\mathsf{Q}\cdot \tilde{\eta} &= -2\sqrt{2}\,\mathcal{D}_m V^m & \tilde{\mathsf{Q}}\cdot \tilde{\eta} &= -2i[B, C] \\
\mathsf{Q}\cdot B &= \sqrt{2}\,\eta & \tilde{\mathsf{Q}}\cdot B &= -\sqrt{2}\,\tilde{\eta} \\
\mathsf{Q}\cdot C &= 0 & \tilde{\mathsf{Q}}\cdot C &= 0 \\
\mathsf{Q}\cdot V_m &= -\sqrt{2}\,\tilde{\psi}_m & \tilde{\mathsf{Q}}\cdot V_m &= -\sqrt{2}\,\psi_m\,.
\end{aligned} \tag{3.7}$$

The associated infinitesimal transformations are given as usual by $\delta_Q X \equiv \epsilon\mathsf{Q}\cdot X$ and $\delta_{\tilde{\mathsf{Q}}} X \equiv \tilde{\epsilon}\tilde{\mathsf{Q}}\cdot X$, where ϵ and $\tilde{\epsilon}$ are the corresponding anti-commuting scalar parameters. The BRST operators Q and $\tilde{\mathsf{Q}}$ are nilpotent and anticommute with each other, up to gauge transformations and on-shell (utilising the $\chi^\pm, \eta$ and $\tilde{\eta}$ equations of motion from (3.1)). The on-shell condition may be removed by introducing auxiliary fields in a standard way [17].

The other symmetries of the theory include the following. First we have the twisted rotations, which act in the usual way:

$$\begin{aligned}
\mathbb{M}_{mn}\cdot A_p &= (x_m\partial_n - x_n\partial_m)A_p + \delta_{mp}A_n - \delta_{np}A_m \\
\mathbb{M}_{mn}\cdot \psi_p &= (x_m\partial_n - x_n\partial_m)\psi_p + \delta_{mp}\psi_n - \delta_{np}\psi_m \\
\mathbb{M}_{mn}\cdot \tilde{\psi}_p &= (x_m\partial_n - x_n\partial_m)\tilde{\psi}_p + \delta_{mp}\psi_n - \delta_{np}\tilde{\psi}_m \\
\mathbb{M}_{mn}\cdot \chi^\pm_{pq} &= (x_m\partial_n - x_n\partial_m)\chi^\pm_{pq} - \delta_{mp}\chi^\pm_{nq} - \delta_{np}\chi^\pm_{mq} + \delta_{mq}\chi^\pm_{pn} - \delta_{nq}\chi^\pm_{pm} \\
\mathbb{M}_{mn}\cdot \eta &= (x_m\partial_n - x_n\partial_m)\eta \\
\mathbb{M}_{mn}\cdot \tilde{\eta} &= (x_m\partial_n - x_n\partial_m)\tilde{\eta} \\
\mathbb{M}_{mn}\cdot B &= (x_m\partial_n - x_n\partial_m)B \\
\mathbb{M}_{mn}\cdot C &= (x_m\partial_n - x_n\partial_m)C \\
\mathbb{M}_{mn}\cdot V_p &= (x_m\partial_n - x_n\partial_m)V_p + \delta_{mp}V_n - \delta_{np}V_m\,.
\end{aligned} \tag{3.8}$$

The spacetime translations act in the standard way,

$$\mathsf{P}_m\cdot\Phi = \partial_m\Phi \tag{3.9}$$

for all fields Φ.

The R-symmetry generators $\mathsf{p}_m, \mathsf{k}_m$ lead to the following symmetries of the twisted theory:

$$\begin{aligned}
\mathsf{p}_m\cdot\psi_p &= -\frac{1}{2}(4\chi^-_{mp} + \delta_{mp}\tilde{\eta}) \\
\mathsf{p}_m\cdot\tilde{\psi}_p &= \frac{1}{2}(4\chi^+_{mp} + \delta_{mp}\eta) \\
\mathsf{p}_m\cdot C &= 2V_m \\
\mathsf{p}_m\cdot V_p &= -\delta_{mp}B\,,
\end{aligned} \tag{3.10}$$

and, using δ_{k} defined by $\delta_{\mathsf{k}} X = \kappa^m \mathsf{k}_m \cdot X$,

$$\begin{aligned}
\delta_{\mathsf{k}}\chi^+_{pq} &= -2(\kappa_{[p}\tilde{\psi}_{q]})^+ \\
\delta_{\mathsf{k}}\chi^-_{pq} &= 2(\kappa_{[p}psi_{q]})^- \\
\delta_{\mathsf{k}}\eta &= -2\kappa^m\tilde{\psi}_m \\
\delta_{\mathsf{k}}\tilde{\eta} &= 2\kappa^m\psi_m \\
\delta_{\mathsf{k}}B &= 2\kappa^m V_m \\
\delta_{\mathsf{k}}V_p &= -\kappa_p C\,.
\end{aligned} \tag{3.11}$$

The original $\mathcal{N}=4$ theory was conformally invariant, but the twisted theory is invariant under scale transformations generated by D, but not under the conformal boosts in which K_m acts in the standard way [9], so that in this sense it is scale invariant but not conformally invariant. The trace of the stress tensor is non-zero, but is a total derivative, so that the integral over $\mathbb{R}^4$ of the trace of the stress tensor vanishes (with suitable boundary conditions), signalling dilatation invariance. However, we have found some modifications of the action of the action of K_m that are a symmetry of the action (3.1). These are conveniently written using δ_{K}, defined by $\delta_{\mathsf{K}} X = \kappa^m \mathsf{K}_m \cdot X$. One finds

$$\begin{aligned}
\delta_{\mathsf{K}}A_p &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)A_p + 4\kappa_{[p}x_{q]}A^q \\
\delta_{\mathsf{K}}\psi_p &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 3x_m)\psi_p + 4(\kappa_{[p}x_{q]})^-\psi^q \\
\delta_{\mathsf{K}}\tilde{\psi}_p &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 3x_m)\tilde{\psi}_p + 4(\kappa_{[p}x_{q]})^+\tilde{\psi}^q \\
\delta_{\mathsf{K}}\chi^+_{pq} &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)\chi^+_{pq} - 4(\kappa_{[p}x^k\chi^+_{q]k})^+ - (\kappa_{[p}x_{q]})^+\eta \\
\delta_{\mathsf{K}}\chi^-_{pq} &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)\chi^-_{pq} - 4(\kappa_{[p}x^k\chi^-_{q]k})^- - (\kappa_{[p}x_{q]})^-\tilde{\eta} \\
\delta_{\mathsf{K}}\eta &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 3x_m)\eta + 4\kappa^m x^n\chi^+_{mn} \\
\delta_{\mathsf{K}}\tilde{\eta} &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 3x_m)\tilde{\eta} + 4\kappa^m x^n\chi^-_{mn} \\
\delta_{\mathsf{K}}B &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)B \\
\delta_{\mathsf{K}}C &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)C \\
\delta_{\mathsf{K}}V_p &= \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)V_p.
\end{aligned} \tag{3.12}$$

These are not standard conformal boost transformations due to the presence of extra terms, some of which mix different fields.

The algebra of these symmetries does not close to give the conformal algebra (2.14), and in particular the commutator of P and K is not of the right form. However, from the discussion of the previous section, we are interested in the full $\mathrm{Spin}(5,1)$ twisting in which the full conformal group is twisted with the full R-symmetry group, and in particular this means twisting P with p and K with k. We therefore consider the symmetries generated by

$$\begin{aligned}
\mathbb{P}_m &\equiv \mathsf{P}_m + \mu\mathsf{p}_m \\
\mathbb{K}_m &\equiv \mathsf{K}_m + \mu^{-1}\mathsf{k}_m\,,
\end{aligned} \tag{3.13}$$

where we have included an arbitrary parameter μ. Notice that the scalings $\mathsf{p}\to\mu\mathsf{p}, \mathsf{k}\to\frac{1}{\mu}\mathsf{k}$ leave invariant the $\mathrm{SO}(5,1)$ R-symmetry algebra generated by $\{\mathsf{p},\mathsf{k},\mathsf{m},\mathsf{d}\}$.

For these new translations we thus have

$$
\begin{aligned}
\mathbb{P}_m \cdot A_p &= \partial_m A_p \\
\mathbb{P}_m \cdot \psi_p &= \partial_m \psi_p - \frac{1}{2}\mu(4\chi^-_{mp} + \delta_{mp}\tilde{\eta}) \\
\mathbb{P}_m \cdot \tilde{\psi}_p &= \partial_m \tilde{\psi}_p + \frac{1}{2}\mu(4\chi^+_{mp} + \delta_{mp}\eta) \\
\mathbb{P}_m \cdot \chi^\pm_{pq} &= \partial_m \chi^\pm_{pq} \\
\mathbb{P}_m \cdot \eta &= \partial_m \eta \\
\mathbb{P}_m \cdot \tilde{\eta} &= \partial_m \tilde{\eta} \\
\mathbb{P}_m \cdot B &= \partial_m B \\
\mathbb{P}_m \cdot C &= \partial_m C + 2\mu V_m \\
\mathbb{P}_m \cdot V_p &= \partial_m V_p - \mu\delta_{mp} B \,,
\end{aligned} \tag{3.14}
$$

and for the new special conformal transformations, using $\delta_{\mathbb{K}}$ defined by $\delta_{\mathbb{K}} X = \kappa^m \mathbb{K}_m \cdot X$, we have

$$
\begin{aligned}
&\delta_{\mathbb{K}} A_p = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)A_p + 4\kappa_{[p}x_{q]}A^q \\
&\delta_{\mathbb{K}} \psi_p = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 3x_m)\psi_p + 4(\kappa_{[p}x_{q]})^-\psi^q \\
&\delta_{\mathbb{K}} \tilde{\psi}_p = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 3x_m)\tilde{\psi}_p + 4(\kappa_{[p}x_{q]})^+\tilde{\psi}^q \\
&\delta_{\mathbb{K}} \chi^+_{pq} = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)\chi^+_{pq} - 4(\kappa_{[p}x^k\chi^+_{q]k})^+ - (\kappa_{[p}x_{q]})^+\eta - 2\mu^{-1}(\kappa_{[p}\tilde{\psi}_{q]})^+ \\
&\delta_{\mathbb{K}} \chi^-_{pq} = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)\chi^-_{pq} - 4(\kappa_{[p}x^k\chi^-_{q]k})^- - (\kappa_{[p}x_{q]})^-\tilde{\eta} + 2\mu^{-1}(\kappa_{[p}\psi_{q]})^- \\
&\delta_{\mathbb{K}} \eta = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 3x_m)\eta + 4\kappa^m x^n\chi^+_{mn} - 2\mu^{-1}\kappa^m\tilde{\psi}_m \\
&\delta_{\mathbb{K}} \tilde{\eta} = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 3x_m)\tilde{\eta} + 4\kappa^m x^n\chi^-_{mn} + 2\mu^{-1}\kappa^m\psi_m \\
&\delta_{\mathbb{K}} B = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)B + 2\mu^{-1}\kappa^m V_m \\
&\delta_{\mathbb{K}} C = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)C \\
&\delta_{\mathbb{K}} V_p = \kappa^m(2x_m x\cdot\partial - x^2\partial_m + 2x_m)V_p - \mu^{-1}\kappa_p C \,.
\end{aligned} \tag{3.15}
$$

One can check that the generators $\{\mathbb{P}_m, \mathbb{M}_{mn}, \mathbb{D}, \mathbb{K}_m\}$ as defined above are symmetries of the action (3.1) and satisfy the Spin$(5,1)$ algebra (2.14) for any μ.

Just as twisting the ordinary supersymmetries of the $\mathcal{N}=4$ theory gives a number of fermionic generators, including the two scalar BRST charges $\mathsf{Q}, \tilde{\mathsf{Q}}$, twisting the conformal supersymmetries also gives a set of fermionic symmetries including those generated by two further scalar charges S and $\tilde{\mathsf{S}}$. We will call these *SBRST* generators. They generate the following symmetries of the action (3.1)

$$
\begin{aligned}
\mathsf{S}\cdot A_m &= -x^n(4\chi^+_{mn} - \delta_{mn}\eta) \\
\mathsf{S}\cdot \psi_m &= 2x^n F^-_{mn} - ix_m[B,C] + 4ix^n[V_m,V_n]^+ \\
\mathsf{S}\cdot \tilde{\psi}_m &= \sqrt{2}[4x^n(\mathcal{D}_{[m}V_{n]})^- - 2x_{[m}\mathcal{D}^n V_{n]} + (x^n\mathcal{D}_n + 2)V_m] \\
\mathsf{S}\cdot \chi^+_{mn} &= -2\sqrt{2}(x_{[m}\mathcal{D}_{n]})^+ B \\
\mathsf{S}\cdot \chi^-_{mn} &= 4i[(x_{[m}V_{n]})^-, B]
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{S}\cdot\eta &= -2\sqrt{2}(x^n\mathcal{D}_n+2)B \\
\mathsf{S}\cdot\tilde{\eta} &= -4i[x^nV_n,B] \\
\mathsf{S}\cdot B &= 0 \\
\mathsf{S}\cdot C &= -2\sqrt{2}x^n\psi_n \\
\mathsf{S}\cdot V_m &= \frac{1}{\sqrt{2}}x^n(4\chi^-_{mn}+\delta_{mn}\tilde{\eta})
\end{aligned}
\tag{3.16}
$$

and

$$
\begin{aligned}
\tilde{\mathsf{S}}\cdot A_m &= x^n(4\chi^-_{mn}-\delta_{mn}\tilde{\eta}) \\
\tilde{\mathsf{S}}\cdot \psi_m &= \sqrt{2}[4x^n(\mathcal{D}_{[m}V_{n]})^+ - 2x_{[m}\mathcal{D}^nV_{n]} + (x^n\mathcal{D}_n+2)V_m] \\
\tilde{\mathsf{S}}\cdot \tilde{\psi}_m &= -2x^nF^+_{mn} + ix_m[B,C] - 4ix^n[V_m,V_n]^- \\
\tilde{\mathsf{S}}\cdot \chi^+_{mn} &= 4i[(x_{[m}V_{n]})^+,B] \\
\tilde{\mathsf{S}}\cdot \chi^-_{mn} &= 2\sqrt{2}(x_{[m}\mathcal{D}_{n]})^-B \\
\tilde{\mathsf{S}}\cdot \eta &= -4i[x^nV_n,B] \\
\tilde{\mathsf{S}}\cdot \tilde{\eta} &= 2\sqrt{2}(x^n\mathcal{D}_n+2)B \\
\tilde{\mathsf{S}}\cdot B &= 0 \\
\tilde{\mathsf{S}}\cdot C &= 2\sqrt{2}x^n\tilde{\psi}_n \\
\tilde{\mathsf{S}}\cdot V_m &= \frac{1}{\sqrt{2}}x^n(4\chi^+_{mn}+\delta_{mn}\eta)\,.
\end{aligned}
\tag{3.17}
$$

The corresponding infinitesimal transformations are given as usual by $\delta_S X = \xi\mathsf{S}\cdot X$ and $\delta_{\tilde{S}}X = \tilde{\xi}\tilde{\mathsf{S}}\cdot X$, with ξ and $\tilde{\xi}$ the fermionic scalar parameters.

The brackets between all the scalar supercharges $\mathsf{Q},\tilde{\mathsf{Q}},\mathsf{S},\tilde{\mathsf{S}}$ vanish (on-shell and up to gauge transformations) except for

$$
[\mathsf{Q},\mathsf{S}] = -4\mathbb{D}\,, \qquad [\tilde{\mathsf{Q}},\tilde{\mathsf{S}}] = -4\mathbb{D}\,. \tag{3.18}
$$

The SO(4) twisted theory with action (3.1) also has a discrete $\mathbb{Z}_2$ symmetry which acts on both the fields and the coupling constants in the following way. It leaves A_m, B, C, e invariant. It reverses orientation: $\epsilon_{mnpq}\mapsto -\epsilon_{mnpq}$, so that $\theta\mapsto-\theta$. Defining

$$
\tau := \frac{4\pi}{e^2} - i\frac{\theta}{2\pi}\,, \tag{3.19}
$$

we have that $\tau\mapsto\bar{\tau}$. On the remaining fields and constants, it acts as follows:

$$
\begin{aligned}
\chi^\pm_{mn} &\mapsto -\chi^\mp_{mn} \\
(\psi_m,\tilde{\psi}_n) &\mapsto (-\tilde{\psi}_m,-\psi_n) \\
(\eta,\tilde{\eta}) &\mapsto (-\tilde{\eta},-\eta) \\
V_m &\mapsto -V_m \\
\mu &\mapsto -\mu\,.
\end{aligned}
\tag{3.20}
$$

This symmetry leaves the action (3.1) invariant and it exchanges the scalar (S)BRST generators $(\mathsf{Q},\mathsf{S})\mapsto(\tilde{\mathsf{Q}},\tilde{\mathsf{S}})$.

Although the theory (3.1) is conformally invariant and has four (S)BRST symmetries, it is not the case that the (S)BRST supercharges are themselves conformally invariant, i.e., they are not singlets under the action of the SO(5, 1) generated by $\mathbb{M}, \mathbb{P}, \mathbb{D}, \mathbb{K}$. This is clear from (3.18) and the fact that $\mathbb{D}$ is not central in the conformal algebra. Nevertheless the two linear combinations

$$\mathbb{Q} := \mathsf{Q} + \mu\tilde{\mathsf{S}}, \qquad \tilde{\mathbb{Q}} := \tilde{\mathsf{Q}} - \mu\mathsf{S}\,, \tag{3.21}$$

satisfy

$$\mathbb{Q}^2 = 0, \qquad \tilde{\mathbb{Q}}^2 = 0, \qquad \{\mathbb{Q}, \tilde{\mathbb{Q}}\} = 0 \tag{3.22}$$

(on-shell and up to gauge transformations). We will call these scalar supercharges *CBRST* charges (the C denoting 'conformal'). These charges commute with the conformal generators and are mapped into each other by the $\mathbb{Z}_2$ symmetry above. They are the two scalar supercharges of the group theoretical analysis in section 2; see (2.3).

4. Redefining the fields

In the previous section we have shown that the SO(4) twisted theory with action (3.1) is conformally invariant and moreover that there are two CBRST symmetries commuting with the conformal generators. Nevertheless, the action of the conformal generators does not take the standard form — for example, the fields do not transform conventionally under the translations $\mathbb{P}_m$. It is clear that this must occur in general in situations like this where one twists the ordinary translations with a corresponding internal symmetry. This unconventional behaviour under the translations $\mathbb{P}_m$ will cause complications, for example when considering defining the theory on a curved manifold. It is thus convenient, after having done the twisting as in the previous section, to *undo* the R-symmetry translations p_m by redefining all fields Φ according to

$$\Phi \to (e^{-\mu x^m \mathsf{p}_m}) \cdot \Phi\,. \tag{4.1}$$

Explicitly, this gives the following field redefinitions

$$\begin{aligned} V_m &\mapsto V_m + \mu x_m B \\ C &\mapsto C - 2\mu x^m V_m - \mu^2 x^2 B \\ \psi_m &\mapsto \psi_m - \frac{1}{2}\mu x^n (4\chi^-_{mn} - \delta_{mn}\tilde{\eta}) \\ \tilde{\psi}_m &\mapsto \tilde{\psi}_m + \frac{1}{2}\mu x^n (4\chi^+_{mn} - \delta_{mn}\eta)\,. \end{aligned} \tag{4.2}$$

Note that these redefinitions are consistent with the $\mathbb{Z}_2$ symmetry.

It is straightforward to calculate the effect of the $\{\mathbb{M}_{mn}, \mathbb{P}_m, \mathbb{D}\}$ twisted generators on the redefined fields. These are taken as transformations that act on the fields but not on the space-time coordinates x^m. In particular, $\mathbb{P}_m$ does not transform the explicit x^m in the redefinitions (4.2), with the result that its action on the redefined fields is, by construction, now the standard action

$$\mathbb{P}_m \cdot \Phi = \partial_m \Phi \tag{4.3}$$

on any field Φ. Since $\mathbb{M}_{mn}$ and $\mathbb{D}$ commute with $x^m \mathsf{p}_m$, the action of the twisted dilatations and rotations remains standard

$$\begin{aligned} \mathbb{M}_{mn} \cdot \Phi &= (x_m \partial_n - x_n \partial_m + \Sigma_{mn})\Phi \\ \mathbb{D} \cdot \Phi &= (x^n \partial_n + \Delta_\Phi)\Phi\,, \end{aligned} \tag{4.4}$$

where Σ_{mn} are the generators of the spin group in the representation determined by the spin of Φ and Δ_Φ is the twisted conformal weight of Φ. However, the special conformal transformations still contain non-standard μ-dependent terms:

$$\begin{aligned}
&\delta_{\mathbb{K}} A_p = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 2x_m) A_p + 4\kappa_{[p} x_{q]} A^q \\
&\delta_{\mathbb{K}} \psi_p = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 2x_m) \psi_p + 4\kappa_{[p} x_{q]} \psi^q \\
&\delta_{\mathbb{K}} \tilde\psi_p = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 2x_m) \tilde\psi_p + 4\kappa_{[p} x_{q]} \tilde\psi^q \\
&\delta_{\mathbb{K}} \chi^+_{pq} = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 4x_m) \chi^+_{pq} - 8(\kappa_{[p} x^k \chi^+_{q]k})^+ - 2\mu^{-1}(\kappa_{[p}\tilde\psi_{q]})^+ \\
&\delta_{\mathbb{K}} \chi^-_{pq} = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 4x_m) \chi^-_{pq} - 8(\kappa_{[p} x^k \chi^-_{q]k})^- + 2\mu^{-1}(\kappa_{[p}\psi_{q]})^- \\
&\delta_{\mathbb{K}} \eta = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 4x_m) \eta - 2\mu^{-1}\kappa^m \tilde\psi_m \\
&\delta_{\mathbb{K}} \tilde\eta = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 4x_m) \tilde\eta + 2\mu^{-1}\kappa^m \psi_m \\
&\delta_{\mathbb{K}} B = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 4x_m) B + 2\mu^{-1}\kappa^m V_m \\
&\delta_{\mathbb{K}} C = \kappa^m (2x_m x\cdot\partial - x^2\partial_m) C \\
&\delta_{\mathbb{K}} V_p = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 2x_m) V_p + 4\kappa_{[p} x_{q]} V^q - \mu^{-1}\kappa_p C\,.
\end{aligned} \tag{4.5}$$

These transformations differ from the usual transformations

$$\delta_{\mathbb{K}} \Phi = \kappa^m (2x_m x\cdot\partial - x^2\partial_m + 2x_m\Delta_\Phi + 2x^n\Sigma_{mn})\Phi \tag{4.6}$$

by the addition of extra terms proportional to μ^{-1}. These μ^{-1} terms imply the loss of special conformal symmetry in the $\mu \to 0$ limit.

In terms of the redefined fields the CBRST symmetry remains simple:

$$\begin{aligned}
&\mathbb{Q}\cdot A_m = 2\psi_m && \tilde{\mathbb{Q}}\cdot A_m = -2\tilde\psi_m \\
&\mathbb{Q}\cdot \psi_m = \sqrt{2}\,\mathcal{D}_m C && \tilde{\mathbb{Q}}\cdot \psi_m = -2i[V_m, C] \\
&\mathbb{Q}\cdot \tilde\psi_m = -2i[V_m, C] && \tilde{\mathbb{Q}}\cdot \tilde\psi_m = -\sqrt{2}\,\mathcal{D}_m C \\
&\mathbb{Q}\cdot \chi^+_{mn} = -F^+_{mn} + 2i[V_m, V_n]^+ && \tilde{\mathbb{Q}}\cdot \chi^+_{mn} = 2\sqrt{2}\,(\mathcal{D}_{[m} V_{n]})^+ \\
&\mathbb{Q}\cdot \chi^-_{mn} = 2\sqrt{2}\,(\mathcal{D}_{[m} V_{n]})^- && \tilde{\mathbb{Q}}\cdot \chi^-_{mn} = F^-_{mn} - 2i[V_m, V_n]^- \\
&\mathbb{Q}\cdot \eta = 2i[B, C] && \tilde{\mathbb{Q}}\cdot \eta = -2\sqrt{2}\,(\mathcal{D}_m V^m + 2\mu B) \\
&\mathbb{Q}\cdot \tilde\eta = -2\sqrt{2}\,(\mathcal{D}_m V^m + 2\mu B) && \tilde{\mathbb{Q}}\cdot \tilde\eta = -2i[B, C] \\
&\mathbb{Q}\cdot B = \sqrt{2}\,\eta && \tilde{\mathbb{Q}}\cdot B = -\sqrt{2}\,\tilde\eta \\
&\mathbb{Q}\cdot C = 0 && \tilde{\mathbb{Q}}\cdot C = 0 \\
&\mathbb{Q}\cdot V_m = -\sqrt{2}\,\tilde\psi_m && \tilde{\mathbb{Q}}\cdot V_m = -\sqrt{2}\,\psi_m\,,
\end{aligned} \tag{4.7}$$

the only change in comparison with the transformations on the original fields (3.7) being the explicit μ-dependent terms in $\mathbb{Q}\cdot\tilde\eta$ and $\tilde{\mathbb{Q}}\cdot\eta$.

The field redefinition (4.2) takes the action (3.1) to

$$
\begin{aligned}
\mathcal{S}^{(1)} =& 2\pi\tau k + \frac{1}{e^2}\int d^4x \,\mathrm{Tr}\Big(-\mathcal{D}_m B \mathcal{D}^m C - \mathcal{D}_m V_n \mathcal{D}^m V^n - \frac{1}{2}F^+_{mn}F^{+mn} \\
&+ \mathcal{D}_m\psi_n(4\chi^{+mn} - \delta^{mn}\eta) + \mathcal{D}_m\tilde{\psi}_n(4\chi^{-mn} - \delta^{mn}\tilde{\eta}) - \\
&- \frac{i}{8\sqrt{2}}\Big((4\chi^+_{mn} - \delta_{mn}\eta)[4\chi^{+mn} - \delta^{mn}\eta, C] + \\
&\qquad + (4\chi^-_{mn} - \delta_{mn}\tilde{\eta})[4\chi^{-mn} - \delta^{mn}\tilde{\eta}, C]\Big) - \\
&- i\sqrt{2}\Big((4\chi^+_{mn} - \delta_{mn}\eta)[\tilde{\psi}^m, V^n] - (4\chi^-_{mn} - \delta_{mn}\tilde{\eta})[\psi^m, V^n]\Big) + \\
&+ i\sqrt{2}(\psi_m[\psi^m, B] + \tilde{\psi}_m[\tilde{\psi}^m, B]) - \\
&- \frac{1}{2}[B, C]^2 + 2[B, V_m][C, V^m] + \\
&+ [V_m, V_n][V^m, V^n] + 4\mu V^m \mathcal{D}_m B - 2\mu\tilde{\eta}\eta - 4\mu^2 B^2\Big), \qquad (4.8)
\end{aligned}
$$

where k is the instanton number

$$
k := \frac{1}{32\pi^2}\int d^4x \; \mathrm{Tr} * F_{mn}F^{mn}\,. \qquad (4.9)
$$

The redefinitions produce the additional terms proportional to μ and μ^2 in the above action; the explicit x-dependence in the redefinitions (4.2) drops out from the action, and hence it is manifestly invariant under the translations. The modified action (4.8) is invariant under the standard $\mathbb{P}, \mathbb{M}, \mathbb{D}$ transformations (4.3), (4.4), the modified conformal boosts, $\mathbb{K}$ (4.5), and the CBRST transformations (4.7). The generators $\mathbb{P}, \mathbb{M}, \mathbb{D}, \mathbb{K}$ satisfy the SO(5,1) algebra (2.14). The CBRST transformations square to zero, anticommute with each other and commute with these SO(5,1) symmetries.

The change in the F_{mn} terms from (3.1) to (4.8) is obtained by decomposing F_{mn} into its self-dual and anti self-dual parts, then recasting the topological instanton number term (with parameter θ) in terms of the modular parameter τ defined in (3.19). The action (4.8) can then be conveniently expressed (using the $\chi^\pm_{mn}, \eta$ and $\tilde{\eta}$ equations of motion) as a CBRST exact term (or an anti-CBRST exact term), plus a term depending only upon τ and the instanton number k:

$$
\mathcal{S}^{(1)} = \mathbb{Q}\cdot\Psi^{(1)} + 2\pi\tau k = \tilde{\mathbb{Q}}\cdot\tilde{\Psi}^{(1)} - 2\pi\bar{\tau}k\,, \qquad (4.10)
$$

where

$$
\begin{aligned}
\Psi^{(1)} = \frac{1}{e_0^2}\int d^4x \; \mathrm{Tr}\Big(&\frac{1}{2}(F^+_{mn} - 2i[V_m, V_n]^+)\chi^{+mn} - \sqrt{2}(\mathcal{D}_{[m}V_{n]})^-\chi^{-mn} + \frac{1}{\sqrt{2}}\mu B\tilde{\eta} - \\
&- \frac{1}{2\sqrt{2}}V^m\mathcal{D}_m\tilde{\eta} + \frac{1}{\sqrt{2}}B(\mathcal{D}_m\psi^m + i\sqrt{2}\,[\tilde{\psi}_m, V^m]) + \frac{i}{4}\eta[B, C]\Big)\,,
\end{aligned}
\qquad (4.11)
$$

and

$$\tilde{\Psi}^{(1)} = \frac{1}{e_0^2}\int d^4x\ \mathrm{Tr}\Big(-\frac{1}{2}(F^-_{mn} - 2i[V_m, V_n]^-)\chi^{-mn} - \sqrt{2}(\mathcal{D}_{[m}V_{n]})^+\chi^{+mn} + \frac{1}{\sqrt{2}}\mu B\eta - \\ - \frac{1}{2\sqrt{2}}V^m\mathcal{D}_m\eta - \frac{1}{\sqrt{2}}B(\mathcal{D}_m\tilde{\psi}^m - i\sqrt{2}\,[\psi_m, V^m]) - \frac{i}{4}\tilde{\eta}[B, C]\Big)\,. \tag{4.12}$$

The topological observables will be discussed in section 6.

5. Conserved currents

We have seen that the action for the twisted theory in $\mathbb{R}^4$ is CBRST exact (modulo the instanton number term). We will now discuss the conformal currents of the theory, and the derivation of the appropriate flat-space energy-momentum tensor.

We begin by calculating the canonical energy-momentum tensor T^{c}_{mn}, defined in general by

$$T^{\mathrm{c}}_{mn} := \mathrm{Tr}\left(\sum_\Phi \partial_n\Phi\cdot\Pi_m(\Phi) - \delta_{mn}\mathcal{L}\right) \qquad \text{where} \qquad \Pi_m(\Phi) := \partial\mathcal{S}/\partial(\partial^m\Phi) \tag{5.1}$$

for an action of the form $\mathcal{S} = \int d^4x\,\mathrm{Tr}\,\mathcal{L}$ with lagrangian $\mathcal{L}(\Phi, \mathcal{D}_m\Phi)$ and some set of fields $\{\Phi\}$. By construction, this tensor is conserved (on-shell), $\partial^m T^{\mathrm{c}}_{mn} = 0$. Noether's theorem implies that for every symmetry of the theory there exists a conserved current. For conformal symmetries, these conserved currents are suitable moments of an improved energy momentum tensor. This tensor is obtained by adding *improvement terms* to T^{c}_{mn}. A discussion of this can be found in [18]. The purpose of the first such improvement term is to symmetrise the energy-momentum tensor. This first additional term is given by $\partial^p X_{pmn}$, where

$$X_{pmn} := \frac{1}{2}\sum_\Phi \mathrm{Tr}\left((\Sigma_{mn}\Phi)\cdot\Pi_p - (\Sigma_{pm}\Phi)\cdot\Pi_n - (\Sigma_{pn}\Phi)\cdot\Pi_m\right), \tag{5.2}$$

and Σ_{mn} are the usual spin generators. The expression $\partial^p X_{pmn}$ is automatically conserved as a result of the $(m \leftrightarrow p)$ antisymmetry in X_{pmn}, and when added to T^{c}_{mn} constructs what is called the *Belinfante tensor*

$$T^{\mathrm{B}}_{mn} \equiv T^{\mathrm{c}}_{mn} + \partial^p X_{pmn}\,, \tag{5.3}$$

which is symmetric and conserved on-shell. Evaluating this object for the twisted theory (4.8) gives

$$T^{\mathrm{B}}_{mn} = \mathrm{Tr}\Big(-2\mathcal{D}_{(m}B\mathcal{D}_{n)}C - 2\mathcal{D}_mV_p\mathcal{D}_nV^p - F_{mp}F^p_n + \\ + 2\mathcal{D}_p[(\mathcal{D}_{(m}V^p)V_{n)} - (\mathcal{D}_{(m}V_{n)})V^p] + 4\mu V_{(m}\mathcal{D}_{n)}B - \\ - 8\chi^{+\ p}_m(\mathcal{D}_{[n}\psi_{p]})^- - 4i\sqrt{2}\,([\chi^+_{p(m}, V_{n)}]\tilde{\psi}^p + 2\chi^{+\ p}_m[V_{[n}, \tilde{\psi}_{p]}]^-) - \\ - 8\chi^{-\ p}_m(\mathcal{D}_{[n}\tilde{\psi}_{p]})^+ + 4i\sqrt{2}\,([\chi^-_{p(m}, V_{n)}]\psi^p + 2\chi^{-\ p}_m[V_{[n}, \psi_{p]}]^+) - \\ - 2(\mathcal{D}_{(m}\eta)\psi_{n)} - i\sqrt{2}\,[\eta, V_{(m}]\tilde{\psi}_{n)} - 2(\mathcal{D}_{(m}\tilde{\eta})\tilde{\psi}_{n)} + i\sqrt{2}\,[\tilde{\eta}, V_{(m}]\psi_{n)} + \\ + 4i\sqrt{2}\,\chi^{+\ p}_m[\chi^+_{np}, C] + 2i\sqrt{2}\,\psi_m[\psi_n, B] + \delta_{mn}\mathcal{D}_p(\eta\psi^p) + \\ + 4i\sqrt{2}\,\chi^{-\ p}_m[\chi^-_{np}, C] + 2i\sqrt{2}\,\tilde{\psi}_m[\tilde{\psi}_n, B] + \delta_{mn}\mathcal{D}_p(\tilde{\eta}\tilde{\psi}^p) - \delta_{mn}\mathcal{L}^{(1)}\Big)\,. \tag{5.4}$$

The second improvement term is specific for conformally symmetric theories and requires the calculation of the *field-virial* $\mathcal{V}_m$, defined as

$$\mathcal{V}_m \equiv \mathrm{Tr}\left((\delta_{mn}\Delta_\Phi + \Sigma_{mn})\,\Phi\ \cdot \Pi^n\right), \tag{5.5}$$

where Δ_Φ is the conformal weight of Φ. It can then be shown that the condition of special conformal invariance requires that

$$\mathcal{V}_m = \partial^n \sigma_{mn}\,, \tag{5.6}$$

for some function of the fields σ_{mn}.

For the twisted theory (4.8) this condition is satisfied, with

$$\begin{aligned}\sigma_{mn} = \mathrm{Tr}\big(&-2\delta_{mn}BC - 2V_mV_n - \delta_{mn}V^2 + \mu^{-1}(V_m\mathcal{D}_nC - (\mathcal{D}_nV_m)C) -\\ &- 2\mu^{-1}\tilde{\psi}_{(m}\psi_{n)} + \mu^{-1}\delta_{mn}\tilde{\psi}_p\psi^p - \mu^{-1}\epsilon_{mnpq}\tilde{\psi}^p\psi^q\big)\,,\end{aligned} \tag{5.7}$$

and hence

$$\mathcal{V}_m = -2B(\mathcal{D}_mC) + 8\mu V_mB + 2(\mathcal{D}_nV^n)V_m - 4(\mathcal{D}_{(m}V_{n)})V^n - 2\eta\psi_m - 2\tilde{\eta}\tilde{\psi}_m\,, \tag{5.8}$$

where we used the twisted conformal weights in this calculation. Next, the prescription is to take the symmetric object

$$s_{mn} := \sigma_{(mn)}\,, \tag{5.9}$$

and use it to construct

$$Y_{pqmn} = \delta_{pq}s_{mn} + \delta_{mn}s_{pq} - \delta_{pm}s_{qn} - \delta_{pn}s_{qm} - \frac{1}{3}(\delta_{pq}\delta_{mn} - \delta_{pm}\delta_{qn})s^k_k\,. \tag{5.10}$$

The second addition to the energy-momentum tensor is then $\frac{1}{2}\partial^p\partial^q Y_{pqmn}$, which on its own is both symmetric and conserved. Thus, for flat-space theories with full conformal invariance, the final improved form of the energy-momentum tensor T_{mn} is given by

$$T_{mn} := T^{\mathrm{B}}_{mn} + \frac{1}{2}\partial^p\partial^q Y_{pqmn}\,. \tag{5.11}$$

This fully improved energy-momentum tensor T_{mn} is also traceless, as required in order to construct conformal currents as the moments $J_{im} = k_i^n T_{mn}$, using the usual 15 conformal Killing vectors k_i^n $(i = 1, \ldots, 15)$ in flat space.

For the twisted theory with action (4.8), this second improvement term is explicitly given by

$$\begin{aligned}\frac{1}{2}\partial^p\partial^q Y_{pqmn} = \mathrm{Tr}\Bigg(&(\delta_{mn}\mathcal{D}^2 - \mathcal{D}_{(m}\mathcal{D}_{n)})\times\\
&\times\left[-\frac{2}{3}BC - \frac{1}{6}\mu^{-1}(V^k\mathcal{D}_kC - (\mathcal{D}_kV^k)C) + \frac{2}{3}\mu^{-1}\tilde{\psi}^k\psi_k\right] +\\
&+ \mathcal{D}^2\left[-V_mV_n + \frac{1}{2}\mu^{-1}(V_{(m}\mathcal{D}_{n)}C - (\mathcal{D}_{(m}V_{n)})C) - \mu^{-1}\tilde{\psi}_{(m}\psi_{n)}\right] +\\
&+ \delta_{mn}\mathcal{D}^p\mathcal{D}^q\left[-V_pV_q + \frac{1}{2}\mu^{-1}(V_{(p}\mathcal{D}_{q)}C - (\mathcal{D}_{(p}V_{q)})C) - \mu^{-1}\tilde{\psi}_{(p}\psi_{q)}\right] -\\
&- \mathcal{D}_m\mathcal{D}^p\left[-V_nV_p + \frac{1}{2}\mu^{-1}(V_{(n}\mathcal{D}_{p)}C - (\mathcal{D}_{(n}V_{p)})C) - \mu^{-1}\tilde{\psi}_{(n}\psi_{p)}\right] -\\
&- \mathcal{D}_n\mathcal{D}^p\left[-V_mV_p + \frac{1}{2}\mu^{-1}(V_{(m}\mathcal{D}_{p)}C - (\mathcal{D}_{(m}V_{p)})C) - \mu^{-1}\tilde{\psi}_{(m}\psi_{p)}\right]\Bigg)\,.\end{aligned} \tag{5.12}$$

This expression is to be added to the expression in (5.4) to give the final traceless, symmetric, conserved energy-momentum tensor $T_{mn} := T^{\mathrm{B}}_{mn} + \frac{1}{2}\partial^p\partial^q Y_{pqmn}$ for the theory (4.8). This improved energy-momentum tensor T_{mn} is CBRST exact, with $T_{mn} = \mathbb{Q} \cdot G_{mn}$ for a certain function G_{mn} of the fields. This is given by the flat space specialisation of the result in the next section.

6. Coupling to gravity

We now turn to the formulation of the topological theory on a riemannian 4-manifold $\mathcal{M}$ with metric g_{mn}, and seek a BRST-invariant action which is invariant under Weyl scalings and which reduces to the flat space theory considered in previous sections. The energy-momentum tensor will be defined as usual in terms of the action $\mathcal{S}$ by

$$T_{mn} = \frac{2}{\sqrt{g}} \frac{\delta \mathcal{S}}{\delta g^{mn}} \,. \tag{6.1}$$

The first step to coupling a flat-space gauge invariant theory to a general curved background is with a minimal prescription of covariantising the action via $\delta_{mn} \mapsto g_{mn}$, $d^4x \mapsto \sqrt{g}\, d^4x$ and $\mathcal{D} \mapsto \nabla \equiv \mathcal{D} + \Gamma$ (where $\mathcal{D}$ is the gauge covariant derivative and $\Gamma(g)$ is the Levi-Civita connection on $\mathcal{M}$). In general, if this is done the resulting energy-momentum tensor, as defined by (6.1), will reduce in flat space to the Belinfante tensor (5.3). (Note that fields satisfying a metric-dependent constraint, such as the fields $\chi^{\pm}_{mn}$ when defined on a curved manifold, will also transform under metric variations (see [4]).) For conformal theories in flat space one needs to add a second improvement term in the energy-momentum tensor, giving (5.11). The corresponding curved space theory must include additional non-minimal terms which reproduce this second improvement term $\frac{1}{2}\partial^p\partial^q Y_{pqmn}$ in the flat-space case. The extra terms required are

$$\mathcal{S}^{\partial\partial Y} := \int_{\mathcal{M}} d^4x \sqrt{g} \left(\frac{1}{2} R_{mn} s^{mn} - \frac{1}{12} R s^k_k \right), \tag{6.2}$$

where R_{mn} and R are the Ricci tensor and scalar of $\mathcal{M}$ while s_{mn} is the minimally coupled version of the expression given by (5.9) and (5.7) in the previous section.

However, in the case of the theory with action (4.8), it turns out that defining such a theory in curved space via minimal coupling and adding the above terms does *not* produce a topological conformal field theory — for example, the resulting energy-momentum tensor defined by (6.1) is not traceless or CBRST exact in curved space. One is, however, free to add further curvature dependent terms to this action in order to obtain if possible a curved space theory which has the required properties and reduces in flat space to the theory given in section 4. It does turn out to be possible to find such additional terms, and this curved space conformal topological quantum field theory will now be presented and discussed.

The action for this conformal TQFT is given by

$$\mathcal{S} = \mathcal{S}^{m.c.} + \mathcal{S}^{\partial\partial Y} + \mathcal{S}^{CR} + \mathcal{S}^{\mathrm{Euler}} \,, \tag{6.3}$$

where $\mathcal{S}^{m.c.}$ is the minimally-coupled version of the action (4.8), $\mathcal{S}^{\partial\partial Y}$ is the action (6.2) above, and the other two terms on the right-hand side of (6.3) are given by

$$\mathcal{S}^{CR} = -\frac{1}{144\mu^2 e^2}\int_{\mathcal{M}} d^4x\sqrt{g}\ \mathrm{Tr}\, C^2 R^2 \tag{6.4}$$

and

$$\mathcal{S}^{\mathrm{Euler}} = \frac{\pi^2}{e^2\mu^2}\int_{\mathcal{M}} d^4x\ \sqrt{g}\, e(\mathcal{M})\ \mathrm{Tr}\, C^2 \tag{6.5}$$

where

$$e(\mathcal{M}) := \frac{1}{32\pi^2}\left(W_{mnpq}W^{mnpq} - 2R_{mn}R^{mn} + \frac{2}{3}R^2\right) \tag{6.6}$$

is the Euler density on $\mathcal{M}$.

The action (6.3) has the following properties:

- $\mathcal{S}$ is invariant under nilpotent CBRST transformations $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$, with $\{\mathbb{Q},\tilde{\mathbb{Q}}\} = 0$.
- $\mathcal{S}$ is CBRST and anti-CBRST exact.
- The energy-momentum tensor T_{mn} arising from $\mathcal{S}$ is CBRST and anti-CBRST exact.
- T_{mn} is traceless, and the corresponding local Weyl symmetries of $\mathcal{S}$ commute with the CBRST symmetries.

To see these properties, begin by rewriting the action $\mathcal{S}$ of (6.3) in the form

$$\mathcal{S} = \mathcal{S}^{(2)} + \mathcal{S}^{\mathrm{Euler}}, \tag{6.7}$$

with

$$\begin{aligned}
\mathcal{S}^{(2)} =& \frac{1}{e^2}\int_{\mathcal{M}} d^4x\sqrt{g}\,\mathrm{Tr}\Bigg(-\frac{1}{2}(-F^+_{mn} + 2i[V_m,V_n]^+)^2 - 4((\nabla_{[m}V_{n]})^-)^2 - \nabla_m B\nabla^m C - \\
&\qquad - \left(\nabla_m V^m + 2\mu B + \frac{1}{12\mu}CR\right)^2 + \\
&\qquad + \frac{1}{\mu}(V^m\nabla^n C - \tilde{\psi}^m\psi^n)\left(R_{mn} - \frac{1}{3}g_{mn}R\right) + \\
&\qquad + \nabla_m\psi_n(4\chi^{+mn} - g^{mn}\eta) + \nabla_m\tilde{\psi}_n(4\chi^{-mn} - g^{mn}\tilde{\eta}) - 2\mu\tilde{\eta}\eta - \\
&\qquad - \frac{i}{8\sqrt{2}}\big((4\chi^+_{mn} - g_{mn}\eta)[4\chi^{+mn} - g^{mn}\eta, C] + \\
&\qquad\qquad + (4\chi^-_{mn} - g_{mn}\tilde{\eta})[4\chi^{-mn} - g^{mn}\tilde{\eta}, C]\big) - \\
&\qquad - i\sqrt{2}\big((4\chi^+_{mn} - g_{mn}\eta)[\tilde{\psi}^m, V^n] - (4\chi^-_{mn} - g_{mn}\tilde{\eta})[\psi^m, V^n]\big) + \\
&\qquad + i\sqrt{2}\,(\psi_m[\psi^m, B] + \tilde{\psi}_m[\tilde{\psi}^m, B]) - \frac{1}{2}[B,C]^2 + 2[B,V_m][C,V^m]\Bigg) + \\
&+ 2\pi\tau k\,,
\end{aligned} \tag{6.8}$$

and $\mathcal{S}^{\mathrm{Euler}}$ given in (6.5).

The curved space CBRST symmetries of the action (6.7) are given by (∇_m is the gauge and diffeomorphism covariant derivative)

$$
\begin{aligned}
&\mathbb{Q}\cdot A_m = 2\psi_m && \tilde{\mathbb{Q}}\cdot A_m = -2\tilde{\psi}_m \\
&\mathbb{Q}\cdot \psi_m = \sqrt{2}\,\nabla_m C && \tilde{\mathbb{Q}}\cdot \psi_m = -2i[V_m, C] \\
&\mathbb{Q}\cdot \tilde{\psi}_m = -2i[V_m, C] && \tilde{\mathbb{Q}}\cdot \tilde{\psi}_m = -\sqrt{2}\,\nabla_m C \\
&\mathbb{Q}\cdot \chi^+_{mn} = -F^+_{mn} + 2i[V_m, V_n]^+ && \tilde{\mathbb{Q}}\cdot \chi^+_{mn} = 2\sqrt{2}\,(\nabla_{[m}V_{n]})^+ \\
&\mathbb{Q}\cdot \chi^-_{mn} = 2\sqrt{2}\,(\nabla_{[m}V_{n]})^- && \tilde{\mathbb{Q}}\cdot \chi^-_{mn} = F^-_{mn} - 2i[V_m, V_n]^- \\
&\mathbb{Q}\cdot \eta = 2i[B, C] && \tilde{\mathbb{Q}}\cdot \eta = -2\sqrt{2}\Big(\nabla_m V^m + 2\mu B + \frac{1}{12\mu}CR\Big) \\
&\mathbb{Q}\cdot \tilde{\eta} = -2\sqrt{2}\Big(\nabla_m V^m + 2\mu B + \frac{1}{12\mu}CR\Big) && \tilde{\mathbb{Q}}\cdot \tilde{\eta} = -2i[B, C] \\
&\mathbb{Q}\cdot B = \sqrt{2}\,\eta && \tilde{\mathbb{Q}}\cdot B = -\sqrt{2}\,\tilde{\eta} \\
&\mathbb{Q}\cdot C = 0 && \tilde{\mathbb{Q}}\cdot C = 0 \\
&\mathbb{Q}\cdot V_m = -\sqrt{2}\,\tilde{\psi}_m && \tilde{\mathbb{Q}}\cdot V_m = -\sqrt{2}\,\psi_m \\
&\mathbb{Q}\cdot g_{mn} = 0 && \tilde{\mathbb{Q}}\cdot g_{mn} = 0\,.
\end{aligned}
\tag{6.9}
$$

The terms $\mathcal{S}^{(2)}$ and $\mathcal{S}^{\mathrm{Euler}}$ are separately invariant. These CBRST transformations differ from the minimally-coupled version of the flat-space transformations (4.7) by the addition of CR terms to $\mathbb{Q}\cdot\tilde{\eta}$ and $\tilde{\mathbb{Q}}\cdot\eta$. Since C and the metric g_{mn} are invariant under $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$, these terms do not affect the calculation of anticommutators of the CBRST transformations, and indeed one can readily show that $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ square to zero and anti-commute with each other, up to a gauge transformation and using the equations of motion following from the action (6.3).

The action (6.7) is not only CBRST and anti-CBRST invariant, but is also exact. To see this, first note that $\mathcal{S}^{(2)}$ can be written in the CBRST exact forms:

$$
\begin{aligned}
\mathcal{S}^{(2)} &= \mathbb{Q}\cdot\Psi^{(2)} + 2\pi\tau k + \text{on-shell terms} \\
&= \tilde{\mathbb{Q}}\cdot\tilde{\Psi}^{(2)} - 2\pi\bar{\tau}k + \text{on-shell terms},
\end{aligned}
\tag{6.10}
$$

where

$$
\begin{aligned}
\Psi^{(2)} = \frac{1}{e^2}\int_{\mathcal{M}} d^4x\sqrt{g}\,\mathrm{Tr}\Big(&\frac{1}{2}(F^+_{mn} - 2i[V_m, V_n]^+)\chi^{+mn} - \sqrt{2}\,(\nabla_{[m}V_{n]})^-\chi^{-mn} + \\
&+ \frac{1}{2\sqrt{2}}\left(\nabla_m V^m + 2\mu B + \frac{1}{12\mu}CR\right)\tilde{\eta} + \\
&+ \frac{1}{\sqrt{2}\,\mu}V^m\psi^n\left(R_{mn} - \frac{1}{3}g_{mn}R\right) + \\
&+ \frac{1}{\sqrt{2}}B\big(\nabla_m\psi^m + i\sqrt{2}\,[\tilde{\psi}_m, V^m]\big) + \frac{i}{4}\eta[B, C]\Big)
\end{aligned}
\tag{6.11}
$$

and

$$\tilde{\Psi}^{(2)} = \frac{1}{e^2}\int_{\mathcal{M}} d^4x\sqrt{g}\,\mathrm{Tr}\biggl(-\frac{1}{2}(F^-_{mn} - 2i[V_m, V_n]^-)\chi^{-mn} - \sqrt{2}\,(\nabla_{[m}V_{n]})^+\chi^{+mn} + \\ + \frac{1}{2\sqrt{2}}\left(\nabla_m V^m + 2\mu B + \frac{1}{12\mu}CR\right)\eta - \\ - \frac{1}{\sqrt{2}\,\mu}V^m\tilde{\psi}^n\left(R_{mn} - \frac{1}{3}g_{mn}R\right) - \\ - \frac{1}{\sqrt{2}}B\bigl(\nabla_m\tilde{\psi}^m - i\sqrt{2}\,[\psi_m, V^m]\bigr) - \frac{i}{4}\tilde{\eta}[B,C]\biggr)\,. \tag{6.12}$$

Furthermore, locally the Euler density is a total derivative and can be written as the divergence of some Z_m:

$$e(\mathcal{M}) = \frac{1}{32\pi^2}\left(W_{mnpq}W^{mnpq} - 2R_{mn}R^{mn} + \frac{2}{3}R^2\right) = \nabla^m Z_m\,. \tag{6.13}$$

It is then easy to see that the lagrangian $\mathcal{L}^{\mathrm{Euler}}$ for the action $\mathcal{S}^{\mathrm{Euler}} = \int_{\mathcal{M}} d^4x\,\sqrt{g}\mathcal{L}^{\mathrm{Euler}}$ is also exact up to derivative terms:

$$\mathcal{L}^{\mathrm{Euler}} = \mathbb{Q}\Lambda + \mathbb{V} = \tilde{\mathbb{Q}}\tilde{\Lambda} + \mathbb{V}\,, \tag{6.14}$$

where

$$\Lambda := -\sqrt{2}\frac{\pi^2}{e^2\mu^2}\,Z^m\,\mathrm{Tr}(C\psi_m)\,, \\ \tilde{\Lambda} := \sqrt{2}\frac{\pi^2}{e^2\mu^2}\,Z^m\,\mathrm{Tr}(C\tilde{\psi}_m)\,, \tag{6.15}$$

and

$$\mathbb{V} := \frac{\pi^2}{e^2\mu^2}\,\nabla_m\left(Z^m\,\mathrm{Tr}\,C^2\right)\,. \tag{6.16}$$

Although Z_m is not globally well-defined, any two choices of Z_m will differ by a globally well-defined vector field, and the change in Z_m under a diffeomorphism will be similarly well-defined, so that the ambiguity in the action $\mathcal{S}^{\mathrm{Euler}}$ will be the sum of a surface term (which will vanish for compact $\mathcal{M}$ or with suitable boundary conditions) and an exact term (which will not contribute to the functional integral).

The next step is to show exactness of the energy-momentum tensor of the theory defined by the action (6.3), with CBRST symmetries (6.9). First consider the action $\mathcal{S}^{(2)}$ of (6.8) (a similar argument to the following appears in a related context in [19]. See also [20] for a discussion of topological field theory in the BV formalism, which includes an analysis of metric dependence in the BRST transformations). Varying (6.8) with respect to the metric we find

$$\delta_g\mathcal{S}^{(2)} = \mathbb{Q}\cdot(\delta_g\Psi^{(2)}) + [\delta_g, \mathbb{Q}]\Psi^{(2)} + \delta_g\{\text{on-shell terms}\} \\ = \mathbb{Q}\cdot\left(\delta_g\Psi^{(2)} + \frac{1}{2e^2\sqrt{2}}\int_{\mathcal{M}} d^4x\sqrt{g}\,\,\mathrm{Tr}\,\delta_g(\nabla_m V^m)\tilde{\eta}\right) + (\text{on-shell terms})\,. \tag{6.17}$$

(The $\delta_g\{$on-shell terms$\}$ in the first line in (6.17) involve the $\tilde{\eta}$ and $\chi^{\pm}$ equations of motion. The metric variations of the $\chi^{\pm}$ equations of motion only generate further equations of motion terms. The metric variation of the $\tilde{\eta}$ equation of motion term however generates a term which does not vanish on-shell. However, this term combines with a second term to give the CBRST exact term involving $\nabla_m V^m$ in the second line in (6.17). This second term arises from the commutator $[\delta_g, \mathbb{Q}]$, which only acts non-trivially on the $\tilde{\eta}$ terms in $\Psi^{(2)}$. Alternatively, one can work with the form of the action given below for which the CBRST transformations are nilpotent off-shell, and show that the metric dependence is CBRST exact without using equations of motion.) Now, writing

$$\delta_g \Psi^{(2)} + \frac{1}{2e^2\sqrt{2}} \int_{\mathcal{M}} d^4x \sqrt{g}\, \mathrm{Tr}\, \delta_g(\nabla_m V^m)\tilde{\eta} = \frac{1}{2e^2} \int_{\mathcal{M}} d^4x \sqrt{g}\, \delta g^{mn} G^{(2)}_{mn}\,, \tag{6.18}$$

for some $G^{(2)}_{mn}$ defined by the above relation, it follows that the energy-momentum tensor derived from the action $\mathcal{S}^{(2)}$ is CBRST exact with

$$T^{(2)}_{mn} = \mathbb{Q} \cdot G^{(2)}_{mn} + \text{on-shell terms}\,, \tag{6.19}$$

with the analogous statements holding for the $\mathbb{Z}_2$ related generator $\tilde{\mathbb{Q}}$. The other contribution to the full energy-momentum tensor T_{mn} of the full action (6.7) comes from the action (6.5). Since this action is exact (up to the topological term involving $\mathbb{V}$), and only contains the CBRST inert fields C, g_{mn}, it follows directly that the metric variation of this action is also exact. Thus the full energy-momentum tensor derived from the action (6.3) is CBRST (and anti-CBRST) exact.

The final property of the theory defined by (6.3) is that the trace of the energy-momentum tensor vanishes on-shell. A straightforward calculation based upon the action (6.8) yields the result that the trace of the corresponding energy-momentum tensor is given by

$$g^{mn} T^{(2)}_{mn} = \frac{1}{4\mu^2} \nabla^m \nabla^n \left(\left(R_{mn} - \frac{1}{2} g_{mn} R \right) \mathrm{Tr}(C^2) \right). \tag{6.20}$$

This is precisely cancelled by the trace of the energy-momentum tensor coming from the other part (6.5) of the full action. Then, since the theory with action (6.3) has a traceless energy-momentum tensor on-shell, this action is Weyl invariant under a combination of local Weyl rescalings of the metric $g_{mn} \mapsto \exp(-2w(x))\, g_{mn}$ and the action of the Weyl symmetries on the fields. The latter are given by

$$\begin{aligned}
\delta_W A_m &= 0 & \delta_W \eta &= 2w\eta - \mu^{-1}\tilde{\psi}_m \nabla^m w \\
\delta_W \psi_m &= 0 & \delta_W \tilde{\eta} &= 2w\tilde{\eta} + \mu^{-1}\psi_m \nabla^m w \\
\delta_W \tilde{\psi}_m &= 0 & \delta_W B &= 2wB + \mu^{-1} V_m \nabla^m w \\
\delta_W \chi^+_{mn} &= \mu^{-1}(\tilde{\psi}_{[m}\nabla_{n]} w)^+ & \delta_W C &= 0 \\
\delta_W \chi^-_{mn} &= -\mu^{-1}(\psi_{[m}\nabla_{n]} w)^- & \delta_W V_m &= -\frac{1}{2}\mu^{-1} C \nabla_m w\,.
\end{aligned} \tag{6.21}$$

These curved-space Weyl transformations contain non-conventional $1/\mu$ terms, as one would expect from the presence of such terms in the flat-space conformal transformations. These Weyl symmetries δ_W commute with the CBRST transformations $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$.

On-shell conditions appearing in the curved space formulation in this section can be lifted by writing off-shell formulations using auxiliary fields, just as in the SO(4) twisted theory (see [17]). The auxiliary fields required may be denoted $P, N^{\pm}_{mn}$. The $\mathbb{Z}_2$ symmetry acts on these as $P \to -P, N^{\pm}_{mn} \to -N^{\mp}_{mn}$. The off-shell form of the action (6.8) is

$$\begin{aligned}\mathcal{S}^{(2)} =& \frac{1}{e^2}\int_{\mathcal{M}} d^4x\sqrt{g}\,\mathrm{Tr}\Big(\frac{1}{2}N^{+}_{mn}\big(N^{mn}+2F^{mn}-4i[V^m,V^n]\big)+\\ &+\frac{1}{2}N^{-}_{mn}\big(N^{mn}-4\sqrt{2}\nabla^{[m}V^{n]}\big)+\\ &+\frac{1}{8}P\left(P+4\sqrt{2}\left(\nabla_m V^m+2\mu B+\frac{1}{12\mu}CR\right)\right)+\\ &+\frac{1}{\mu}(V^m\nabla^n C-\tilde{\psi}^m\psi^n)\left(R_{mn}-\frac{1}{3}g_{mn}R\right)-\nabla_m B\nabla^m C+\\ &+\nabla_m\psi_n(4\chi^{+mn}-g^{mn}\eta)+\nabla_m\tilde{\psi}_n(4\chi^{-mn}-g^{mn}\tilde{\eta})-2\mu\tilde{\eta}\eta-\\ &-\frac{i}{8\sqrt{2}}\Big((4\chi^{+}_{mn}-g_{mn}\eta)[4\chi^{+mn}-g^{mn}\eta,C]+\\ &\qquad+(4\chi^{-}_{mn}-g_{mn}\tilde{\eta})[4\chi^{-mn}-g^{mn}\tilde{\eta},C]\Big)-\\ &-i\sqrt{2}\big((4\chi^{+}_{mn}-g_{mn}\eta)[\tilde{\psi}^m,V^n]-(4\chi^{-}_{mn}-g_{mn}\tilde{\eta})[\psi^m,V^n]\big)+\\ &+i\sqrt{2}\big(\psi_m[\psi^m,B]+\tilde{\psi}_m[\tilde{\psi}^m,B]\big)-\frac{1}{2}[B,C]^2+2[B,V_m][C,V^m]\Big)+\\ &+2\pi\tau k\,.\end{aligned} \tag{6.22}$$

The off-shell CBRST symmetries are given by

$$\begin{aligned}
&\mathbb{Q}\cdot A_m = 2\psi_m && \tilde{\mathbb{Q}}\cdot A_m = -2\tilde{\psi}_m\\
&\mathbb{Q}\cdot\psi_m = \sqrt{2}\,\nabla_m C && \tilde{\mathbb{Q}}\cdot\psi_m = -2i[V_m,C]\\
&\mathbb{Q}\cdot\tilde{\psi}_m = -2i[V_m,C] && \tilde{\mathbb{Q}}\cdot\tilde{\psi}_m = -\sqrt{2}\,\nabla_m C\\
&\mathbb{Q}\cdot\chi^{\pm}_{mn} = N^{\pm}_{mn} && \tilde{\mathbb{Q}}\cdot\chi^{+}_{mn} = N^{\pm}_{mn}\\
&\mathbb{Q}\cdot N^{\pm}_{mn} = 2i\sqrt{2}[\chi^{\pm}_{mn},C] && \tilde{\mathbb{Q}}\cdot N^{\pm}_{mn} = 2i\sqrt{2}[\chi^{\pm}_{mn},C]\\
&\mathbb{Q}\cdot\eta = 2i[B,C] && \tilde{\mathbb{Q}}\cdot\eta = P\\
&\mathbb{Q}\cdot\tilde{\eta} = P && \tilde{\mathbb{Q}}\cdot\tilde{\eta} = -2i[B,C]\\
&\mathbb{Q}\cdot P = 2i\sqrt{2}[\tilde{\eta},C] && \tilde{\mathbb{Q}}\cdot P = 2i\sqrt{2}[\eta,C]\\
&\mathbb{Q}\cdot B = \sqrt{2}\,\eta && \tilde{\mathbb{Q}}\cdot B = -\sqrt{2}\,\tilde{\eta}\\
&\mathbb{Q}\cdot C = 0 && \tilde{\mathbb{Q}}\cdot C = 0\\
&\mathbb{Q}\cdot V_m = -\sqrt{2}\,\tilde{\psi}_m && \tilde{\mathbb{Q}}\cdot V_m = -\sqrt{2}\,\psi_m\\
&\mathbb{Q}\cdot g_{mn} = 0 && \tilde{\mathbb{Q}}\cdot g_{mn} = 0\,.
\end{aligned} \tag{6.23}$$

Then in this formulation with these auxiliary fields $\mathbb{Q}^2 = \tilde{\mathbb{Q}}^2 = 0$ up to a gauge transformation with parameter $2\sqrt{2}C$ and $\{\mathbb{Q},\tilde{\mathbb{Q}}\} = 0$, without use of field equations. With these modifications, the action is given by the sum of the topological term $2\pi\tau k$ plus an exact piece, without needing to use field equations, and the metric variation of the action is also exact off-shell. However, the energy momentum tensor is still only traceless on-shell.

Turning to the construction of observables for the conformal TQFT with action (6.3), we first need suitable operators $\mathcal{O}$. The functionals studied in [9] are also suitable here, and the following gauge invariant, CBRST closed (but not exact) expressions arise:

$$
\begin{aligned}
\mathcal{O}_0 &= \int_{\gamma_0} \operatorname{Tr} C^2 \\
\mathcal{O}_1 &= \int_{\gamma_1} \sqrt{2}\,\operatorname{Tr}\,(C \wedge \psi) \\
\mathcal{O}_2 &= \int_{\gamma_2} \frac{1}{2}\operatorname{Tr}\left(\psi \wedge \psi + \frac{1}{2\sqrt{2}} C \wedge F\right) \\
\mathcal{O}_3 &= \int_{\gamma_3} \frac{1}{4}\operatorname{Tr}\,(\psi \wedge F) \\
\mathcal{O}_4 &= \int_{\gamma_4} \frac{1}{32}\operatorname{Tr}\,(F \wedge F)\,,
\end{aligned}
\tag{6.24}
$$

together with a $\mathbb{Z}_2$ related set of observables corresponding to the cohomology of $\tilde{\mathbb{Q}}$. The γ_s, $s = 0, \ldots, 4$ are homology cycles on $\mathcal{M}$ (γ_0 is a point). The operators $\mathcal{O}_k$ and their anti-CBRST analogues satisfy the descent equations given in [9]. The integrands in (6.24) are also invariant under the local scale transformations (6.21). Note that the integrands in any of these observables could be multiplied by any scalar function of the curvature and its derivatives, such as $e(\mathcal{M})$, and would remain CBRST closed but not exact. However, these would only be Weyl-invariant for special combinations of curvatures, such as the square of the Weyl tensor.

7. Theta angle and S-duality

The lorentzian theory has a real theta-term in the action

$$
S_r = -\frac{\theta}{32\pi^2} \int \operatorname{Tr} F \wedge F \tag{7.1}
$$

which is Wick rotated to an imaginary term

$$
S_i = -i\frac{\theta}{32\pi^2} \int \operatorname{Tr} F \wedge F \tag{7.2}
$$

in the euclideanised action. As usual, the parameter θ must be an angle, periodically identified, if the euclideanised path integral is to be single-valued, and the presence of the parameter θ leads to theta-vacua instead of the naive vacuum. However, the euclidean theory is written down directly in euclidean space, and need have nothing to do with any Wick rotation. In particular, if it is not required to be the euclideanisation of a real lorentzian action, there is no reason why the theta-term has to be imaginary, and one could instead write down a real theta-term (7.1) in euclidean space. Indeed, it is natural to have a real action, and this also follows from reduction from 9+1 dimensions [21]. To see this, consider including a term

$$
S = \int C \wedge \operatorname{Tr}(F \wedge F) \tag{7.3}
$$

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in the 9+1 dimensional Yang-Mills action, where C is a background 6-form. Such terms arise for example in considering D5-D9 brane systems. Then reducing on a euclidean 6-torus gives, among other terms, the real theta-term (7.1) in 3+1 dimensions, while reducing on five space and one time dimension again gives a real theta-term, but this time in euclidean space. If theta is real in euclidean space, there is no reason to require theta to be an angle.

In the lorentzian or the euclideanised theory, the angle theta can be combined with the coupling constant e into a complex variable τ which parameterises the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. In the euclidean theory, on the other hand, a real theta term leads to coupling constants parameterising $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$. As will be discussed in the next section, this is the coset structure required by holography, and so the holographic dual should be one in which the theta-term is real, not imaginary.

In the euclidean theory, there are then two choices of action, one with real theta-term and one with an imaginary one. The imaginary one is the one that is conventionally used in topological field theory and is naturally associated with a real lorentzian action. The action with a real theta is the one that is natural from reduction from 9+1 dimensions, and is also the one that is required for holography — the holographic dual of the IIB* theory in de Sitter space is a euclidean super-Yang-Mills theory with a *real* theta-term. The formulae in this paper have all been written for the imaginary case, but it is straightforward to obtain the real case by taking $\theta \to i\theta$ throughout.

The lorentzian Yang-Mills theory has a classical $\mathrm{SL}(2,\mathbb{R})$ symmetry broken to a discrete $\mathrm{SL}(2,\mathbb{Z})$ S-duality symmetry in the quantum theory. The euclidean and euclideanised theories also have a classical $\mathrm{SL}(2,\mathbb{R})$ symmetry. This can be seen by considering the Yang-Mills theory from a dimensional reduction from a 6 dimensional theory, which could be 6-dimensional (1,1) supersymmetric Yang-Mills, or the (2,0) supersymmetric tensor multiplet theory. The lorentzian and euclidean theories can be obtained by reducing from 5+1 dimensions on a euclidean or a lorentzian 2-torus, respectively. Wick rotating 5+1 dimensional (1,1) supersymmetric Yang-Mills to a euclideanised theory in 6 dimensions and then reducing on a 2-torus yields the euclideanised four-dimensional gauge theory. In each case, there is a reduction on a 2-torus and so there is an $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the reduced theory resulting from the large diffeomorphisms on the 2-torus. On truncating to the massless Yang-Mills sector in 4 dimensions, the $\mathrm{SL}(2,\mathbb{Z})$ is enhanced to an $\mathrm{SL}(2,\mathbb{R})$ classical symmetry. In the euclideanised or lorentzian theories, this is then broken to the subgroup $\mathrm{SL}(2,\mathbb{Z})$ preserving the periodic identification of the angle θ. It is natural to conjecture that all three theories in fact have a $\mathrm{SL}(2,\mathbb{Z})$ S-duality symmetry in the full quantum theory, and moreover that the twisted theory also has an S-duality symmetry.

8. Topological holography

The lorentzian $\mathcal{N}=4$ supersymmetric Yang-Mills theory has a dual holographic description as IIB string theory on $\mathrm{AdS}_5 \times S^5$ [3], with the anti-de Sitter space formulation giving a dual description of the Yang-Mills theory at large 't Hooft coupling. Wick-rotating this dual pair, the euclideanised $\mathcal{N}=4$ Yang-Mills theory has a holographic formulation on $H^5 \times S^5$, the euclidean version of $\mathrm{AdS}_5 \times S^5$ with the anti-de Sitter space continued to the hyperbolic space H^5 [22, 1]. It was argued in [1] that the holographic dual of the euclidean $\mathcal{N}=4$ supersymmetric Yang-Mills theory is the IIB* string theory on $\mathrm{dS}_5 \times H^5$, where dS_5 is five dimensional de Sitter space, with the euclidean conformal group $SO(5,1)$ arising as the de Sitter group, and the R-symmetry group arising from the isometries of H^5. These three dualities arise from considering D3-branes, Wick-rotated D3-branes (i.e., instantonic D-branes) and the euclidean E4-branes of [1], respectively. Whereas the D3-branes are timelike 4-surfaces in 9+1 dimensions with a lorentzian super Yang-Mills world-volume theory, the E4-branes are spacelike 4-surfaces in 9+1 dimensions with a euclidean super Yang-Mills world-volume theory.

The euclidean supersymmetric Yang-Mills theory can be twisted in four ways, the A,B and half-twisted models, and the new conformal twisting presented here. It is natural to ask whether these twisted models could still have a holographic description in de Sitter space, or more generally what the dual description of each theory should be at large 't Hooft coupling. One motivation is that the de Sitter formulation could give a useful alternative way of calculating topological invariants. As the 5-dimensional de Sitter symmetry is associated with the conformal symmetry in 4 dimensions, the most promising theory to consider in this context is the conformal twisting, as this is a conformal field theory. In [1], it was proposed that this conformal twisting should indeed have a holographic dual, with correlation functions in the euclidean conformal field theory associated with the partition function of a theory in 5-dimensional de Sitter space.

If J is some composite operator in the Yang-Mills theory, then introducing a source term $\int \phi(x) \cdot J$ with local source $\phi(x)$, which can be thought of as a position-dependent coupling 'constant', then $\left\langle \exp(-\int \phi \cdot J) \right\rangle$ is the generating functional for correlation functions $\langle J(x_1) \cdots J(x_n) \rangle$. This in turn is related, according to the holography conjecture, to the bulk partition function subject to boundary conditions in which the boundary values of certain bulk fields are given by $\phi(x)$, and for many purposes it is useful to think of the Yang-Mills theory as living on that boundary. In the lorentzian case, the boundary is the boundary of AdS_5, and in the euclideanised theory it is the boundary of H^5. For the euclidean case, the situation is more subtle.

The scalar fields in the euclidean Yang-Mills theory take values in the lorentzian space $\mathbb{R}^{5,1}$ and the expectation value of the scalars is a vector in $\mathbb{R}^{5,1}$ which can be spacelike or timelike (or null), and these correspond to different sectors of the theory. This corresponds to whether the separation between the E4-branes that is kept constant in the Maldacena-type limit is spacelike or timelike (or null). For the spacelike separation, the arguments of [1] give a holographic duality between the IIB* theory on $\mathrm{dS}_5 \times H^5$ and a euclidean Yang-Mills theory that lives on the boundary of H^5, while for the timelike separation, the gauge theory lives on a boundary of de Sitter space. This means that the boundary conditions on the boundaries of dS_5 or H^5 are reflected in different sectors of the euclidean conformal field theory [1], a phenomenon that occurs in other examples of holography in which the bulk space is a product of two factors, both of which have a boundary [2, 23].

An important role is played by those operators J which lie in the superconformal current multiplet, the multiplet of the conformal supersymmetry consisting of the the energy-momentum tensor T_{mn} and its superpartners. These couple to fields $\phi(x)$ which lie in the

$\mathcal{N} = 4$ conformal supergravity multiplet, consisting of a background metric g_{mn} coupling to T_{mn}, and its superpartners. In the lorentzian case, this is the standard $\mathcal{N} = 4$ conformal supergravity multiplet of [24], and one-loop quantum corrections induce the conformal supergravity action [25, 26]. The fields in the superconformal gravity multiplet provide the boundary conditions for the bulk fields in the 5-dimensional gauged supergravity multiplet. Euclideanising gives a similar picture, involving the euclideanised conformal gravity and a euclideanised gauged supergravity on H^5.

For the euclidean case, the situation is again similar, with a conformal supergravity multiplet in 4 euclidean dimensions playing a central role. This conformal supergravity arises from dimensional reduction. There is a conformal supergravity in 9+1 dimensions, and reducing on a spatial 6-torus gives the $\mathcal{N} = 4$ conformal supergravity in 3+1 dimensions [24], while reducing on 5 space and one time dimension in a similar manner gives a $\mathcal{N} = 4$ conformal supergravity in 4 euclidean dimensions, with gauge group $\mathrm{SO}(5,1)$ instead of $\mathrm{SO}(6)$. Again, a conformal supergravity action is induced by one-loop corrections. In one sector of the theory, the natural effective supergravity theory consists of fields in dS_5 which are independent of the H^5 coordinates, and in the other sector it consists of fields in H^5 which are independent of the dS_5 coordinates, The gauged supergravity theory in de Sitter space that arises here [1, 23] has gauge group $\mathrm{SO}(5,1)$ and a twisted supersymmetry, and the boundary conditions for these fields are provided by the conformal supergravity fields on the de Sitter boundary. Similarly, there is an $\mathrm{SO}(5,1)$ gauged supergravity on the euclidean space H^5 arising from reducing the IIB* theory on dS_5, and the boundary conditions for these fields are provided by conformal supergravity fields on the boundary of H^5. Both the IIB* theory and the four-dimensional euclidean conformal supergravity have a pair of scalar fields taking values in $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$ instead of the usual $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$. The boundary values of these scalars give rise to the coupling constant and theta parameter in the dual Yang-Mills theory, with the consequence that the theta-term is real, not imaginary, in this case.

We now turn to the question of finding the holographic dual for the twisted gauge theory discussed here. The euclidean Yang-Mills theory has a current supermultiplet $T_{mn}, R_{mA}, \ldots$ where R_{mA} is the supercurrent. The twisting and field redefinitions discussed here then leads to a current multiplet $T_{mn}, R_m, \tilde{R}_m, \ldots$ for the twisted theory, where $R_m, \tilde{R}_m$ are BRST currents. The fields $g_{mn}, \psi_m, \tilde{\psi}_m, \ldots$ coupling to this current multiplet then define a twisted form of the superconformal gravity multiplet, with $\psi_m, \tilde{\psi}_m$ gauge fields for local BRST transformations. These should give the boundary conditions for a 5-dimensional gravity theory in dS_5 or H^5, with fields $g_{MN}, \psi_M, \tilde{\psi}_M, \ldots$ and it is straightforward to read off at least the field content and linearised theory from the structure of the conformal gravity multiplet, as in [25, 26]. This then should constitute the supergravity limit of the holographic dual of the topological conformal field theory. In particular, it is a theory with *local* BRST symmetries. For any bosonic background, and in particular for any manifold, a rigid BRST transformation with constant parameter will be a symmetry — the analogue of the Killing spinor conditions are trivially satisfied — and that rigid BRST symmetry can be used to define a BRST cohomology, giving a gravity theory which is a topological field theory. We will give further details of this construction elsewhere.

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